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Periodic Solutions for a Modified Delayed Heterogeneous Population-Based Model

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Abstract: This paper investigates the existence of periodic solutions for a modified delayed heterogeneous population-based disease model with multiple time delays. We extend the oscillatory result in the literature from a one-delay model to a four-delay model. By means of the mathematical analysis method, we discuss the instability of the unique positive equilibrium point and the boundedness of the solutions. Two sufficient conditions to guarantee the periodic oscillation of the solutions are provided, and computer simulations are given to support the present criteria.

Keywords: disease model, delay, instability, periodic solution

INTRODUCTION

It is known that time delay is very significant in the dynamics of epidemic models as well as population dynamics, which in a model system could cause a stable equilibrium to become unstable and the associated population to become chaotic. Many researchers have studied various delayed mathematical models. For example, Tripathi et al. discussed the dynamics of a predator-prey interaction model with time delay [1]. The stability and bifurcation analysis were investigated in a viral infection model with delays [2-7]. Stability and Hopf bifurcation analysis for delayed SEIRS, SVEIR, and SIQR models [8-11]. In [12], a delayed multi-group SIR epidemic model was studied. Gurbuz et al. were concerned with bifurcation analysis for a delayed dengue fever transmission model [13]. The authors in [14] dealt with global dynamics of a dengue model with multiple delays, incorporating vaccine waning and asymptomatic infection. Guo et al. presented a three-delay dengue model that includes waning vaccine immunity and asymptomatic infections [15]. A novel hybrid fuzzy neural model for analyzing tumor–immune dynamics under uncertainty and time delay was provided in [16]. The authors in [17] have investigated the dynamic behaviors and optimal control for a new delayed epidemic model. Recently, Qi and Zhang provided the following heterogeneous population-based disease model with infection delay [18]:

$$\begin{cases} S'(t) = A - \mu S(t) - \beta e^{-\mu\tau} S(t-\tau)[I_h(t-\tau) + I_s(t-\tau) + I_a(t-\tau)] \\ \qquad + \delta_h I_h(t) + \delta_s I_s(t) + \delta_a I_a(t), \\ I'_h(t) = p_1 \beta e^{-\mu\tau} S(t-\tau)[I_h(t-\tau) + I_s(t-\tau) + I_a(t-\tau)] \\ \qquad - \mu_h I_h(t) - \delta_h I_h(t), \\ I'_s(t) = p_2(1-p_1)\beta e^{-\mu\tau} S(t-\tau)[I_h(t-\tau) + I_s(t-\tau) + I_a(t-\tau)] \\ \qquad - \mu_s I_s(t) - \delta_s I_s(t), \\ I'_a(t) = (1-p_2)(1-p_1)\beta e^{-\mu\tau} S(t-\tau)[I_h(t-\tau) + I_s(t-\tau) + I_a(t-\tau)] \\ \qquad - \mu_a I_a(t) - \delta_a I_a(t), \end{cases} \tag{1}$$

where A is the recruitment rate of susceptible persons, $\mu < \min\{\mu_h, \mu_s, \mu_a\}$ is the natural mortality rate of susceptible persons, $\mu_h, \mu_s, \text{and } \mu_a$ are the total morality rate of I_h, I_s and I_a respectively, β is the infection rate of viral patients and susceptibles, δ_h, δ_s, and δ_a are the cure rate of I_h, I_s and I_a respectively, the combination of represents the proportion of susceptible persons transformed into different types of patients, τ represents the infection delay of the disease in the population. All parameters are nonnegative constants. However, the infection delay for susceptible persons, and verious infected persons of I_h, I_s and I_a may be a different number as the model in [15]. Therefore, in this paper, we extend the model (1) to the following multiple delays:

$$\begin{cases} S'(t) = A - \mu S(t) - \beta e^{-\mu\tau} S(t-\tau_1)[I_h(t-\tau_2) + I_s(t-\tau_3) + I_a(t-\tau_4)] \\ \quad +\delta_h I_h(t) + \delta_s I_s(t) + \delta_a I_a(t), \\ I'_h(t) = p_1 \beta e^{-\mu\tau} S(t-\tau_1)[I_h(t-\tau_2) + I_s(t-\tau_3) + I_a(t-\tau_4)] \\ \quad -\mu_h I_h(t) - \delta_h I_h(t), \\ I'_s(t) = p_2(1-p_1)\beta e^{-\mu\tau} S(t-\tau_1)[I_h(t-\tau_2) + I_s(t-\tau_3) + I_a(t-\tau_4)] \\ \quad -\mu_s I_s(t) - \delta_s I_s(t), \\ I'_a(t) = (1-p_2)(1-p_1)\beta e^{-\mu\tau} S(t-\tau_1)[I_h(t-\tau_2) + I_s(t-\tau_3) + I_a(t-\tau_4)] \\ \quad -\mu_a I_a(t) - \delta_a I_a(t). \end{cases} \tag{2}$$

The initial condition for system (2) is $S(t) = \psi_1(t)$, $I_h(t) = \psi_2(t), I_s(t) = \psi_3(t)$, $I_a(t) = \psi_4(t), t \in [-\max \tau_i, 0]$; $\psi_1(0) > 0, \psi_2(0) > 0, \psi_3(0) > 0, \psi_4(0) > 0.$ Our goal is to discuss the existence of periodic solutions for the system (2). Obviously, one can use the bifurcation method to discuss the existence of a periodic solution. However, the bifurcation method is still hard to deal with a multiple delay system if the delays are different real numbers, as in our simulation.

PRELIMINARIES

For model (2), we have the following lemmas:

Lemma 1 All solutions of the system (2) subject to a non-negative initial condition are bounded.

Proof It is known that time delays do not affect the boundedness of the solutions. Therefore, set time delays are zeros in system (2). Noting that $\mu < \min\{\mu_h, \mu_s, \mu_a\}$, then we have

$$\begin{aligned} S'(t) + I'_h(t) + I'_s(t) + I'_a(t) &= A - \mu S(t) - \mu_h I_h(t) - \mu_s I_s(t) - \mu_a I_a(t) \\ &\le A - \mu[S(t) + I_h(t) + I_s(t) + I_a(t)] \end{aligned} \tag{3}$$

Let $N(t) = S(t) + I_h(t) + I_s(t) + I_a(t)$. From (3) we have

$$N'(t) \le A - \mu N(t) \tag{4}$$

Noting that $-\mu < 0$, from (4) we have $N(t) \le N(0)e^{-\mu t} + \frac{A}{\mu}$. When $t \to +\infty, N(t) \to \frac{A}{\mu}$. This means that all solutions of system (2) are bounded.

Noting that $A > 0$ in system (2), system (2) has a positive equilibrium point. Suppose that $[S^*, {I_h}^*, {I_s}^*, {I_a}^*]^T$ is an positive equilibrium point of the system (2), make the change of the variables $S(t) \to S(t) - S^*$, $I_h(t) \to I_h(t) - {I_h}^*, I_s(t) \to I_s(t) - {I_s}^*, I_a(t) \to I_a(t) - {I_a}^*$, noting that

$$\begin{cases} A-\mu S^*-\beta e^{-\mu\tau}S^*({I_h}^*+{I_s}^*+{I_a}^*)+\delta_h{I_h}^*+\delta_s{I_s}^*+\delta_a{I_a}^*=0, \\ p_1\beta e^{-\mu\tau}S^*({I_h}^*+{I_s}^*+{I_a}^*)-\mu_h{I_h}^*-\delta_h{I_h}^*=0, \\ p_2(1-p_1)\beta e^{-\mu\tau}S^*({I_h}^*+{I_s}^*+{I_a}^*)-\mu_s{I_s}^*-\delta_s{I_s}^*=0, \\ (1-p_2)(1-p_1)\beta e^{-\mu\tau}S^*({I_h}^*+{I_s}^*+{I_a}^*)-\mu_a{I_a}^*-\delta_a{I_a}^*=0 \end{cases} \tag{5}$$

We have the following system

$$\begin{cases} S'(t)=-\mu S(t)+\delta_h I_h(t)+\delta_s I_s(t)+\delta_a I_a(t)-\beta e^{-\mu\tau}({I_h}^*+{I_s}^*+{I_a}^*)S(t-\tau_1) \\ -\beta e^{-\mu\tau}S^*[I_h(t-\tau_2)+I_s(t-\tau_3)+I_a(t-\tau_4)] \\ -\beta e^{-\mu\tau}S(t-\tau_1)[I_h(t-\tau_2)+I_s(t-\tau_3)+I_a(t-\tau_4)], \\ {I'}_h(t)=-\mu_h I_h(t)-\delta_h I_h(t)+p_1\beta e^{-\mu\tau}({I_h}^*+{I_s}^*+{I_a}^*)S(t-\tau_1) \\ +p_1\beta e^{-\mu\tau}S^*[I_h(t-\tau_2)+I_s(t-\tau_3)+I_a(t-\tau_4)] \\ +p_1\beta e^{-\mu\tau}S(t-\tau_1)[I_h(t-\tau_2)+I_s(t-\tau_3)+I_a(t-\tau_4)], \\ {I'}_s(t)=-\mu_s I_s(t)-\delta_s I_s(t)+p_2(1-p_1)\beta e^{-\mu\tau}({I_h}^*+{I_s}^*+{I_a}^*)S(t-\tau_1) \\ +p_2(1-p_1)\beta e^{-\mu\tau}S^*[I_h(t-\tau_2)+I_s(t-\tau_3)+I_a(t-\tau_4)] \\ +p_2(1-p_1)\beta e^{-\mu\tau}S(t-\tau_1)[I_h(t-\tau_2)+I_s(t-\tau_3)+I_a(t-\tau_4)], \\ {I'}_a(t)=-\mu_a I_a(t)-\delta_a I_a(t)+(1-p_2)(1-p_1)\beta e^{-\mu\tau}({I_h}^*+{I_s}^*+{I_a}^*)S(t-\tau_1) \\ +(1-p_2)(1-p_1)\beta e^{-\mu\tau}S^*[I_h(t-\tau_2)+I_s(t-\tau_3)+I_a(t-\tau_4)] \\ +(1-p_2)(1-p_1)\beta e^{-\mu\tau}S(t-\tau_1)[I_h(t-\tau_2)+I_s(t-\tau_3)+I_a(t-\tau_4)]. \end{cases} \tag{6}$$

System (6) can be written in the following matrix form:

$$U'(t)=PU(t)+QU(t-\tau)+g\big(U(t-\tau)\big), \tag{7}$$

Where $U(t)=[S(t),I_h(t),I_s(t),I_a(t)]^T, U(t-\tau)=[S(t-\tau_1),I_h(t-\tau_2),I_s(t-\tau_3),I_a(t-\tau_4)]^T$, $g\big(U(t-\tau)\big)=[-\beta e^{-\mu\tau}S(t-\tau_1)[I_h(t-\tau_2)+I_s(t-\tau_3)+I_a(t-\tau_4)],\ \cdots,$ $(1-p_2)(1-p_1)\beta e^{-\mu\tau}S(t-\tau_1)[I_h(t-\tau_2)+I_s(t-\tau_3)+I_a(t-\tau_4)]\ ]^T$. P and Q both are 4×4 matrices,

$$P=\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & 0 & 0 \\ 0 & 0 & a_{33} & 0 \\ 0 & 0 & 0 & a_{44} \end{pmatrix},$$

$$Q=\begin{pmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{pmatrix}$$

where $a_{11}=-\mu, a_{12}=\delta_h, a_{13}=\delta_s, a_{14}=\delta_a, a_{22}=-\mu_h-\delta_h, a_{33}=-\mu_s-\delta_s, a_{44}=-\mu_a-\delta_a$; $b_{11}=-\beta e^{-\mu\tau}({I_h}^*+{I_s}^*+{I_a}^*), b_{12}=b_{13}=b_{14}=-\beta e^{-\mu\tau}S^*, b_{21}=p_1\beta e^{-\mu\tau}({I_h}^*+{I_s}^*+{I_s}^*), b_{22}=b_{23}=b_{24}=p_1\beta e^{-\mu\tau}S^*,\ b_{31}=p_2(1-p_1)\beta e^{-\mu\tau}({I_h}^*+{I_s}^*+{I_s}^*), b_{32}=b_{33}=b_{34}=p_2(1-p_1)\beta e^{-\mu\tau}S^*,\ b_{41}=(1-p_2)(1-p_1)\beta e^{-\mu\tau}({I_h}^*+{I_s}^*+{I_s}^*), b_{42}=b_{43}=b_{44}=(1-p_2)(1-p_1)\beta e^{-\mu\tau}S^*$. The linearized system of (7) is as follows:

$$U'(t)=PU(t)+QU(t-\tau). \tag{8}$$

Then we have

Lemma 2 If matrix $\Sigma(=P+Q)$ is a nonsingular matrix for selected parameters, then there exists a unique positive equilibrium point $U^*=[S^*,{I_h}^*,{I_s}^*,{I_a}^*]^T$ of the system (2).

Proof Obviously, the zero equilibrium point of the system (8) corresponds to the positive equilibrium point of the system (2). If V^* is an equilibrium point of the system (8), then we have

$$PV^* + QV^* = \sum V^* = 0. \tag{9}$$

According to Cramer's Rule of linear algebraic theory, the system (9) has a unique trivial solution since Q is a nonsingular matrix, namely, $V^* = \mathbf{0}$, implying that the system (2) has a unique positive equilibrium point. The proof is completed.

Based on Lemma 1 and Lemma 2, in the following, we provide two theorems to guarantee the existence of periodic oscillatory solutions.

THE EXISTENCE OF PERIODIC SOLUTIONS

Theorem 1 Assume that zero is the unique equilibrium point of the system (8) for selecting parameter values. Let $\delta_1, \delta_2, \cdots, \delta_4$ be characteristic values of matrix P, and $\theta_1, \theta_2, \cdots, \theta_4$ be characteristic values of matrix Q. If there exists one characteristic value, say δ_1, such that $\delta_1 > 0$, or $Re(\delta_1) > 0$ and $Re(\delta_1) >$ max $\{|\theta_1|, |\theta_2|, \cdots, |\theta_4|\}$. Then the unique trivial solution of the system (8) is unstable, implying that there exists a periodic oscillatory solution in the system (2).

Proof According to the basic differential equation theory, if there exists one characteristic value, say δ_1, such that $\delta_1 > 0$, or $Re(\delta_1) > 0$ and $Re(\delta_1) >$ max $\{|\theta_1|, |\theta_2|, \cdots, |\theta_4|\}$, then the trivial solution $U(t)$ of the system (8) is unstable [13]. Indeed, the characteristic equation associated with the system (8) can be written as follows:

$$\prod_{i=1}^{4}\left(\lambda - \delta_i - \theta_i e^{-\lambda \tau_i}\right) = 0. \tag{10}$$

Therefore, there is a characteristic equation from the system (10) as follows:

$$\lambda - \delta_1 - \theta_1 e^{-\lambda \tau_1} = 0. \tag{11}$$

If $Re(\delta_1) > 0$ and $Re(\delta_1) >$ max $\{|\theta_1|, |\theta_2|, \cdots, |\theta_4|\}$, this means that equation (11) has a positive real part characteristic value. Thus, the trivial solution of the system (8) is unstable. Meanwhile, the nonlinear term of the system (8) is a higher-order infinitesimal as $S(t) \to 0, I_h(t) \to 0, \quad I_s(t) \to 0, I_a(t) \to 0.$ Therefore, the instability of the trivial solution of the system (8) ensures the instability of the trivial solution of the system (7). This means that the unique equilibrium point $U^* = [S^*, {I_h}^*, {I_s}^*, {I_a}^*]^T$ of the system (2) is unstable. The instability of the unique equilibrium point together with the boundedness of the solutions will force the system (2) to generate a limit cycle, namely, there exists a periodic solution of the system (2) [19,20]. The proof is completed.

For simplify, setting $\alpha = \max\{a_{11}, a_{12} + a_{22},\ a_{13} + a_{33},\ a_{14} + a_{44}\}$, $\gamma = \max_{1 \le j \le 4}\left\{b_{jj} + \sum_{i=1, i \ne j}^{4} |b_{ij}|\right\}$. Then we have

Theorem 2 Assume that the conditions of Lemma 1 and Lemma 2 hold. If the following inequality is satisfied

$$\alpha + \gamma > 0 \tag{12}$$

Then the trivial solution of the system (8) is unstable, implying that the system (2) has a periodic solution.

Proof To prove the instability of the trivial solution of the system (8), let $N(t) = S(t) + I_h(t) + I_s(t) + I_a(t)$. So, $N(t) > 0$, and

$$N'(t) \leq \alpha N(t) + \gamma N(t - \tilde{\tau}) \tag{13}$$

Specifically, consider a scalar delayed differential equation

$$y'(t) = \alpha y(t) + \gamma y(t - \tau_0) \tag{14}$$

where $\tau_0 = \min\{\tau_1, \tau_2, \cdots, \tau_4\}$. We have $N(t) \leq y(t)$. Now we prove that the trivial solution of equation (14) is unstable. Indeed, the characteristic equation associated with equation (14) is the following

$$\lambda = \alpha + \gamma e^{-\lambda\tau_0}. \tag{15}$$

We claim that there exists a positive root of (15) under the condition (12). Let $\varphi(\lambda) = \lambda - \alpha - \gamma e^{-\lambda\tau_0}$. Thus, $\varphi(\lambda)$ is a continuous function of λ. When $\lambda = 0$, we have $\varphi(0) = -\alpha - \gamma = -(\alpha + \gamma) < 0$, since $\alpha + \gamma > 0$. On the other hand, there exists a suitably large λ, say λ_0 such that $\varphi(\lambda_0) = \lambda_0 - \alpha - \gamma e^{-\lambda_0\tau_0} > 0$, since $e^{-\lambda_0\tau_0} \rightarrow 0$ as $\lambda_0 \rightarrow +\infty$, Based on the Intermediate Value Theorem, there exists a λ, say $\lambda_1 \in (0, \lambda_0)$ such that $\varphi(\lambda_1) = 0$. In other words, λ_1 is a positive characteristic root of the equation (14). So the trivial solution of the equation (14) is unstable. Since $\tau_0 = \min\{\tau_1, \tau_2, \cdots, \tau_4\}$, this means that for all delays, the trivial equilibrium is unstable, implying that the unique equilibrium point $U^* = [S^*, {I_h}^*, {I_s}^*, {I_a}^*]^T$ of the system (2) is unstable. Similar to Theorem 1, there exists a periodic solution of the system (2). The proof is completed.

SIMULATION RESULT

This simulation is based on the system (2). We first select the parameters as follows: $A = 16, \mu = 0.52, \mu_h = 0.58, \mu_s = 0.62, \mu_a = 0.65, \delta_h = 0.25, \delta_s = 0.38, \delta_a = 0.42, p_1 = 0.55, p_2 = 0.58, \beta = 0.45, \tau = 2.50, \tau_1 = 2.65, \tau_2 = 2.70, \tau_3 = 2.75, \tau_4 = 2.80$, then the unique positive equilibrium point is $[S^*, {I_h}^*, {I_s}^*, {I_a}^*]^T = [9.4439,\ 10.9894, 4.2925, 2.8903]^T$. We see that $a_{11} = -0.52, a_{12} = 0.25,\ a_{13} = 0.38,\ a_{14} = 0.42, a_{22} = -0.83, a_{33} = -1.00, a_{44} = -1.07, b_{11} = -3.3864,\ b_{12} = b_{13} = b_{14} = -1.0739, b_{21} = 1.7272, b_{22} = b_{23} = b_{24} = 0.5909, b_{31} = 0.8196, b_{32} = b_{33} = b_{34} = 0.2803, b_{41} = 0.5935, b_{42} = b_{43} = b_{44} = 0.2031$. Thus, the four eigenvalues of matrix P are $[-0.52, -0.83, -1.00, -1.07]^T$, the four eigenvalues of matrix Q are $[-2.4223, 0.1099, 0.0012, -0.0003]^T$. Noting that all eigenvalues of matrix P are negative, we see that the conditions of Theorem 1 are not satisfied. Since $\alpha = -0.52, \gamma = 2.1443$, the conditions of Theorem 2 are satisfied. Based on Theorem 2, the system (2) has a periodic solution, (see Fig.1). In Fig.2, we only change the parameters $\tau = 2.50, \tau_1 = 2.25, \tau_2 = 2.30, \tau_3 = 2.35, \tau_4 = 2.40$. In Fig.3, we select $A = 15$, time delays are changed to $\tau_1 = 3.25, \tau_2 = 3.30, \tau_3 = 3.35, \tau_4 = 3.40$, the other parameters are the same as in Fig. 2. In Fig.4, we only change $A = 14$, the other parameters are the same as in Fig.3, we see that there exist periodic solutions. Then we select $A = 20, \mu = 0.62, \mu_h = 0.68, \mu_s = 0.72, \mu_a = 0.75, \delta_h = 0.20, \delta_s = 0.25, \delta_a = 0.30, p_1 = 0.85, p_2 = 0.78, \beta = 0.40, \tau = 2.15, \tau_1 = 2.45, \tau_2 = 2.50, \tau_3 = 2.55, \tau_4 = 2.60$, then the unique positive equilibrium point is $[S^*, {I_h}^*, {I_s}^*, {I_a}^*]^T = [9.1454, 18.0507, 2.2381, 0.5776]^T$. We see that $a_{11} = -0.62, a_{12} = 0.20$, $a_{13} = 0.25$, $a_{14} = 0.30, a_{22} = -0.88, a_{33} = -0.97, a_{44} = -1.05, b_{11} = -2.8689$, $b_{12} = b_{13} = b_{14} = -0.9739, b_{21} = 2.4555, b_{22} = b_{23} = b_{24} = 0.8166, b_{31} = 0.3382, b_{32} = b_{33} = b_{34} = 0.1124, b_{41} = 0.0952, b_{42} = b_{43} = b_{44} = 0.0317$. Therefore, $\alpha = -0.62, \gamma = 1.9981$, the conditions of Theorem 2 are satisfied, and there exists a periodic solution (see Fig.5). In Fig.6, we change the time delays as $\tau_1 = 2.20, \tau_2 = 2.25, \tau_3 = 2.30, \tau_4 = 2.35$, and we see that the periodic oscillation is maintained. In Fig.7 and Fig.8, the value of A is changed to $A = 18$, and time delays are changed to $\tau_1 = 3.60, \tau_2 = 3.65, \tau_3 = 3.70, \tau_4 = 3.75$, and $\tau_1 = 4.00, \tau_2 = 4.05, \tau_3 = 4.10, \tau_4 = 4.15$, respectively, the other parameters are the same as in Fig.5, we see that periodic oscillation still occurred.

CONCLUSION

This paper discusses the oscillatory behavior of the solutions for a heterogeneous population-based model with time delays. Based on the method of mathematical analysis, we provided two sufficient conditions to guarantee the oscillation of the solutions. Some numerical simulations are provided to indicate the effectiveness of the criteria. Based on our simulation, the theorem 1 is a stronger sufficient condition than that the theorem 2. The value of A will affect the equilibrium point, and time delays affect the oscillatory frequency.

Competing Interests

The author has declared that no competing interests exist.

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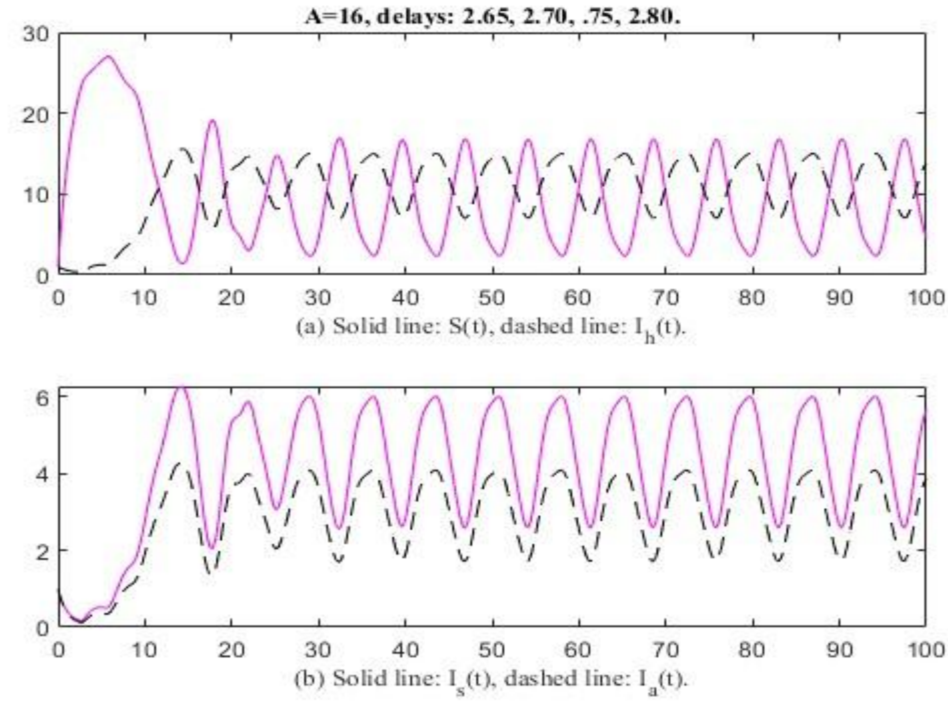

Fig. 1: Oscillation of the solutions, delays: 2.65, 2.70, 2.75, 2.80.

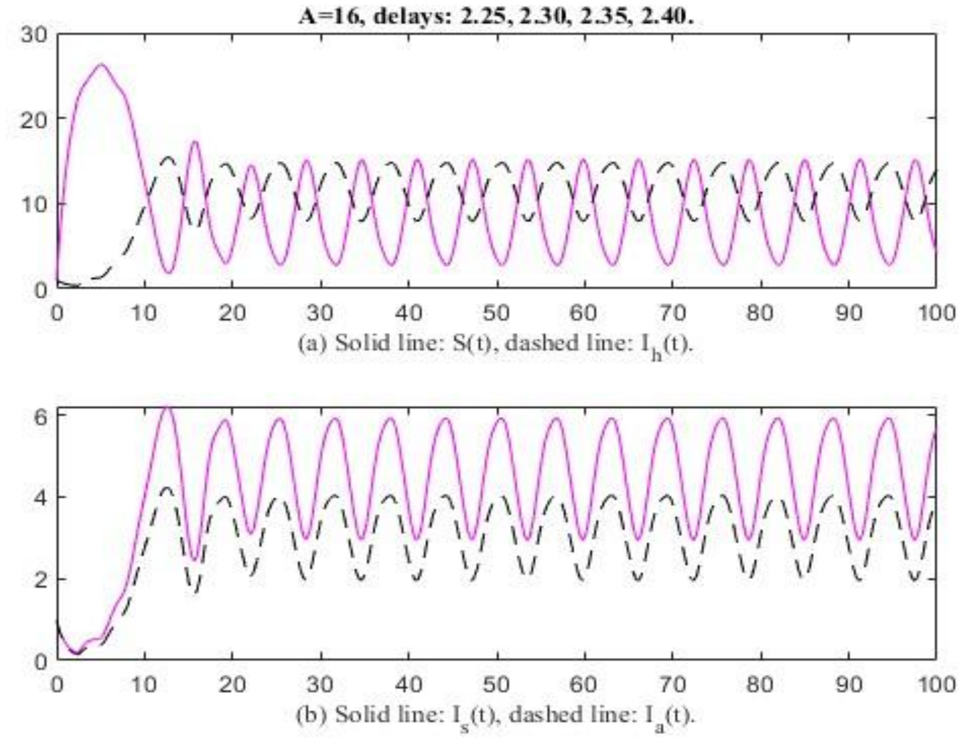

Fig. 2: Oscillation of the solutions, delays: 2.25, 2.30, 2.35, 2.40.

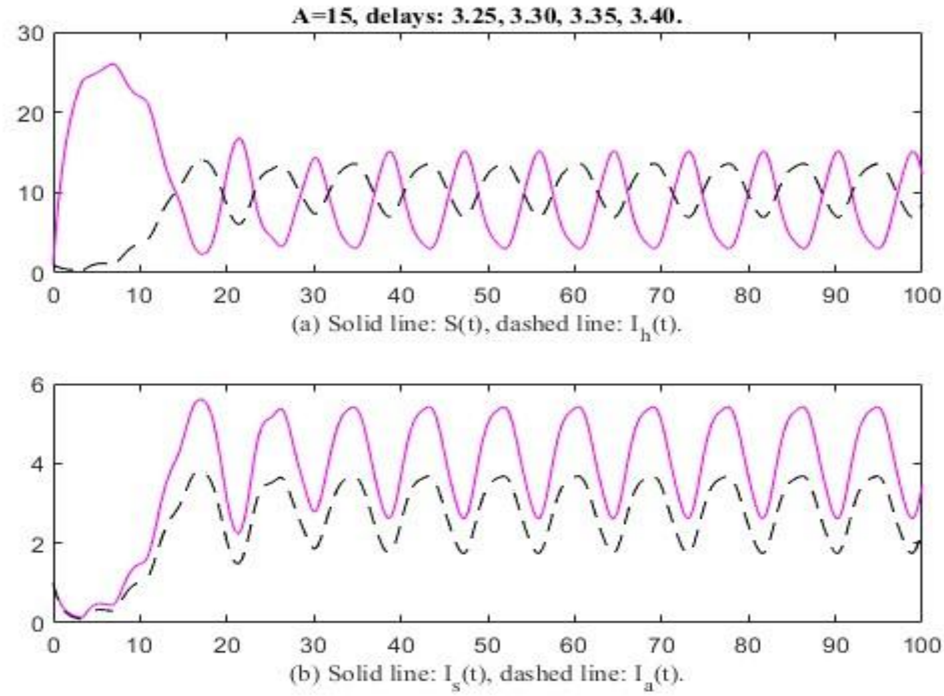

Fig. 3: Oscillation of the solutions, A=15, delays: 3.25, 3.30, 3.35, 3.40.

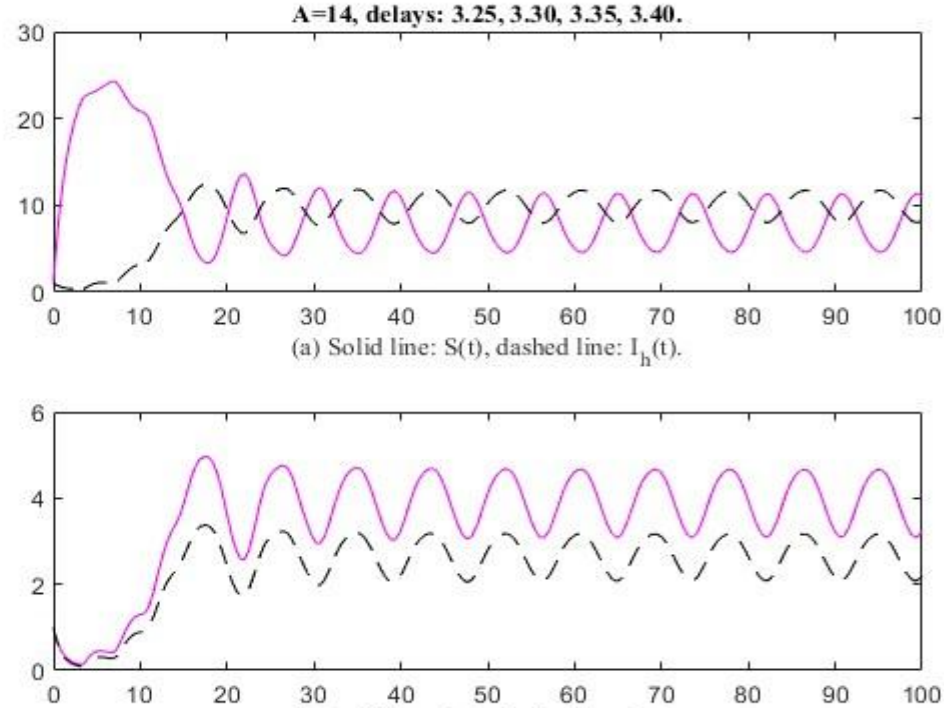

Fig. 4: Oscillation of the solutions, A=14, delays: 3.25, 3.30, 3.35, 3.40.

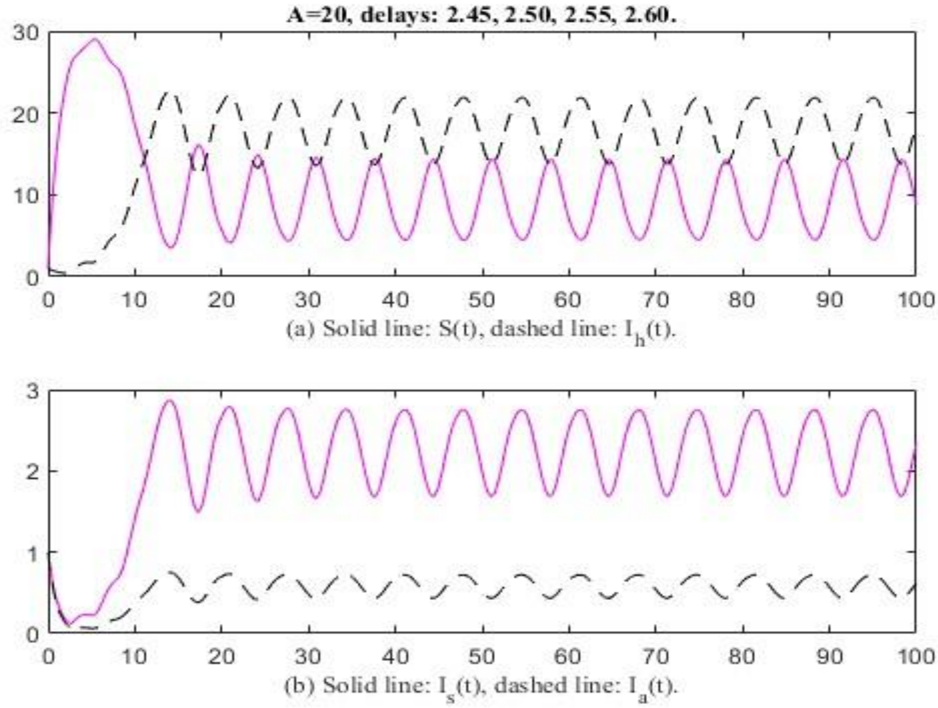

Fig. 5: Oscillation of the solutions, A=20, delays: 2.45, 2.50, 2.55, 2.60.

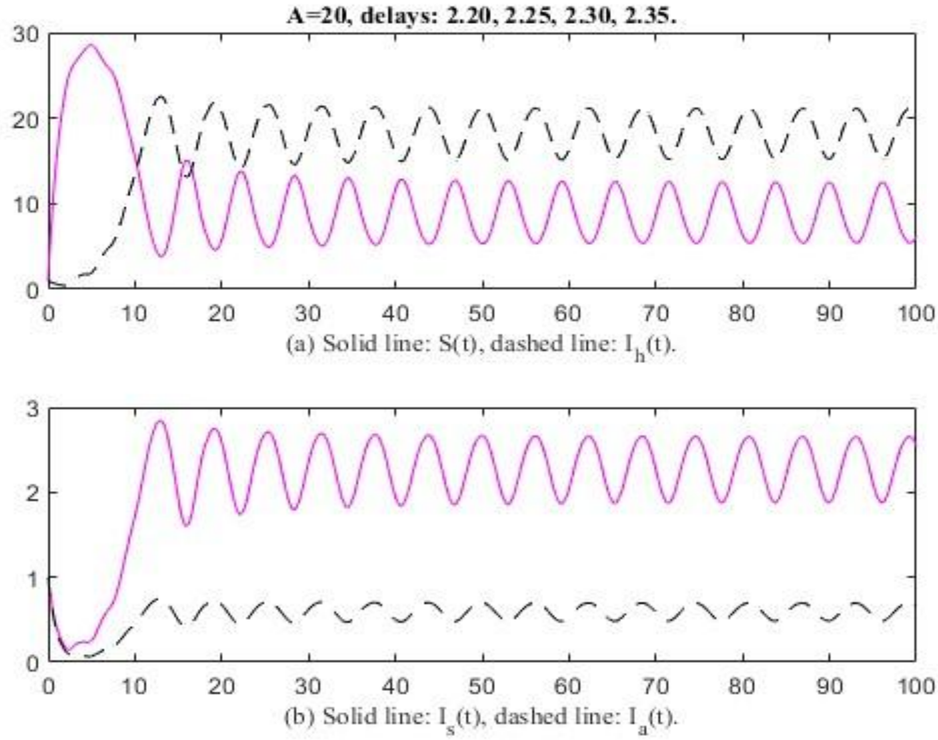

Fig. 6: Oscillation of the solutions, A=20, delays: 2.20, 2.25, 2.30, 2.35.

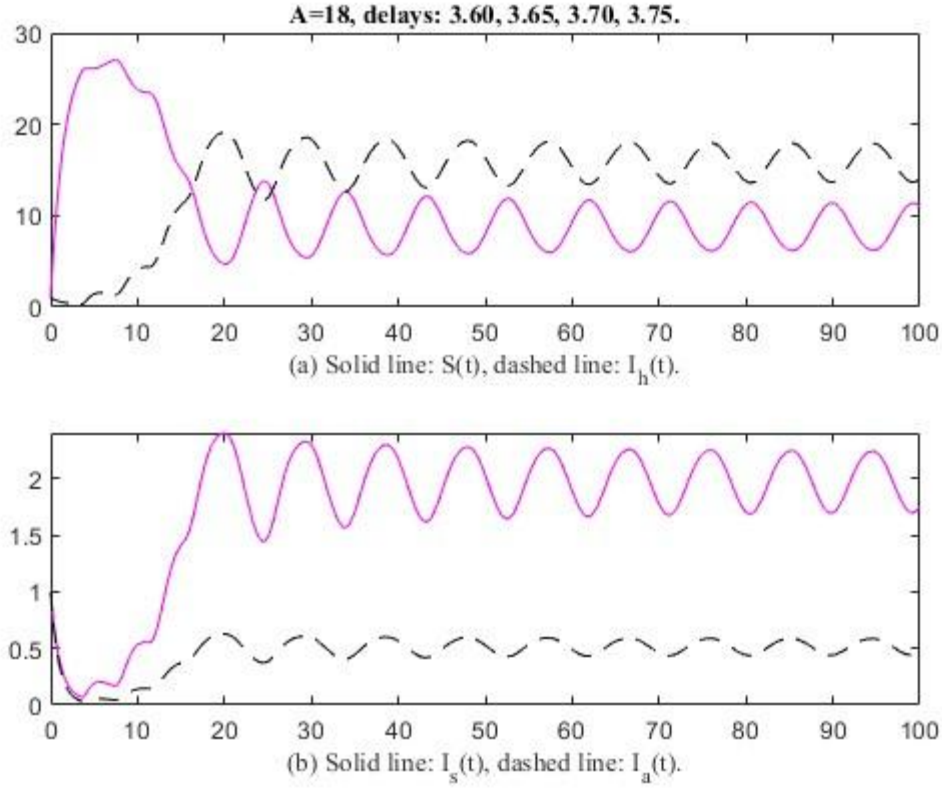

Fig. 7: Oscillation of the solutions, A=18, delays: 3.60, 3.65, 3.70, 3.75.

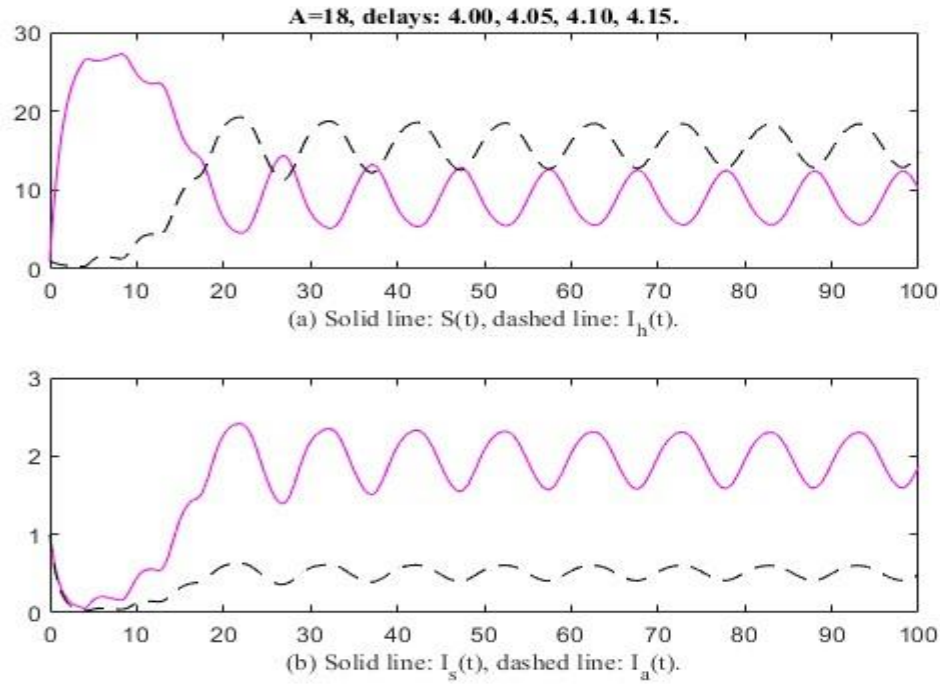

Fig. 8: Oscillation of the solutions, A=18, delays: 4.00, 4.05, 4.10, 4.15.