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SERVICES FOR SCIENCE AND EDUCATION

On the Regularization Method for Solving Ill-Posed Problems with Linear Closed Densely Operators

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ABSTRACT

Let $A: D(A) \subset X \to Y$ be a linear, closed, densely defined unbounded operator, where X and Y are Hilbert spaces. Assume that A is not boundedly invertible. If equation (1) Au=f is solvable, and if $\|f_d - f\| \le d$ then the following results are provided:Problem $F_{a,d}(u) := \|Au - f_d\|^2 + a\|u\|^2$ has a unique global minimizer $u_{a,d}$ for any f_d, and $u_{a,d} = A^*(AA^* + aI_Y)^{-1} f_d$. There is a function α(δ), $\lim_{d \to 0} a(d) = 0$ such that $\lim_{d \to 0} \|u_{a(d),d} - x_0\| = 0$, where x_0 is the unique minimal-norm solution to (1). In this paper we introduce the regularization method solving equation (1) with A being a linear, closed, densely defined unbounded oprator. At the same time give an application to the weak derivative operator equation.

Keywords: Ill-posed problem, regularization method, unbounded linear operator.

INTRODUCTION

Let $A: D(A) \subset X \to Y$ be a linear, closed, densely defined unbounded operator, where X and Y are Hilbert spaces. Consider the equation

$$Au = f \tag{1}$$

Problem finding solution of (1) is called ill-posed in the sense of Hadamard [17] if A is not boundedly invertible. This may happen if the null space $N(A) = \{u: Au = 0\}$ is not trivial, i.e., A is not injective, or if A is injective but A^{-1} is unbounded, i.e., the range of A, $R(A)$ is not closed [3].

If $\|A\| < \infty$, and f_d the noisy data, are given

$$\|f_d - f\| \le d, \tag{2}$$

then problem finding approximate solution of (1) has been extensively studied in the literature in detail (e.g. [1, 2, 3, 8, 15] and references therein).

If A is a linear, closed, densely defined operator or is a linear unbounded operator, problem finding approximate solution of (1) has been some recent research [9, 10, 11, 12, 13, 14, 16], however there are still many open problems such as parameter choice rules of regularization,

Our aim is to study problem finding approximate of (1) when unbounded operator A is a linear, closed, densely defined and the noisy data f_d satisfies (2). We shall introduce the regularization method for solving the problem, a priori and a posteriori parameter choice rule of regularization; at the same time give an application to the weak derivative operator equation.

THE SOME MAIN RESULTS

Lemma 1. [3] Let $A: D(A) \subset X \to Y$ be a linear, closed, densely defined operator, where X and Y are Hilbert spaces, then

1. the operators $T = A^*A$ and $Q = AA^*$ are densely defined, self-adjoint; with A^* be the conjugate operator of A.
2. A^* is closed, densely defined and $A^{**} = A$;
3. the operators $\overset{\circ}{A} := (I_X + A^*A)^{-1}: X \to X$, $A\overset{\circ}{A}: X \to Y$ are both defined on all of X and are bounded, $s\left(\overset{\circ}{A}\right) \in (0,1]$, with I_X be the identity mapping on X. Also, $\overset{\circ}{A}$ is self-adjoint;
4. the operator $\overset{\perp}{A} := (I_Y + AA^*)^{-1}: Y \to Y$ is bounded, self-adjoint and $A^*\overset{\perp}{A}: Y \to X$ is bounded, with I_Y be the identity mapping on Y.

Lemma 2. Let $A: D(A) \subset X \to Y$ be a linear, closed, densely defined operator, where X and Y are Hilbert spaces, if equantion (1) is solvable then minimal-norm solution of (1) is unique, we denote by x_0, furthermore $x_0 \perp N(A)$.

Proof. Because $A: D(A) \subset X \to Y$ is a linear operator, the set of solutions of (1) is a convex set in the Hilbert space X, which we denote by X_0. Furthermore, since A is a closed operator then X_0 is a closed set. Indeed, suppose $x_n \in X_0, x_n \to x$. Because, $x_n \in X_0$, then $Ax_n = f, \forall n$. Therefore, $Ax_n \to f$, then $x \in D(A)$ and $Ax = f$ or $x \in X_0$. So, X_0 is a closed set in Hilbert X. Because X_0 is a convex, closed set in Hilbert X, so it contains only one element with minimal norm x_0 [4] and $x_0 \perp N(A)$ [2].

Theorem 1. For any $f \in Y$, the problem

$$F_a(u) = \|Au - f\|^2 + a\|u\|^2 \to \min, a = \text{const} > 0, \quad (3)$$

has a unique solution $u_a = A^*\left(AA^* + aI_Y\right)^{-1} f$, where I_Y is the identity operator on Y.

Proof. Consider the equation

$$(AA^* + aI_Y)w_a = f, a = \text{const} > 0 \quad (4)$$

which is uniquely solvable

$$w_a = \left(AA^* + aI_Y\right)^{-1} f$$

(Lemma 1).

Let

$$u_a = A^* w_a$$

then

$$\begin{aligned} Au_a &= AA^*\left(AA^* + aI_Y\right)^{-1} f \\ &= \left(AA^* + aI_Y\right)\left(AA^* + aI_Y\right)^{-1} f - aI_Y\left(AA^* + aI_Y\right)^{-1} f \\ &= f - aw_a, \end{aligned}$$

or

$$Au_a - f = -aw_a.$$

We have

$$\begin{aligned} F_a(u+v) &= \|A(u+v) - f\|^2 + a\|u+v\|^2 = \|(Au - f) + Av\|^2 + a\|u+v\|^2 \\ &= \|Au - f\|^2 + \|Av\|^2 + 2\text{Re}\langle Au - f, Av\rangle + a\left(\|u\|^2 + \|v\|^2 + 2\text{Re}\langle u,v\rangle\right) \\ &= \|Au - f\|^2 + \|Av\|^2 + a\|u\|^2 + \|v\|^2 + 2\text{Re}\left(\langle Au - f, Av\rangle + a\langle u,v\rangle\right) \quad (5) \end{aligned}$$

for any $v \in D(A)$. If $u = u_a$, then

$$\langle Au_a - f, Av\rangle + a\langle u_a, v\rangle = -a\langle w_a, Av\rangle + a\langle u_a, v\rangle = -a\langle A^* w_a, v\rangle + a\langle u_a, v\rangle$$
$$= -a\langle u_a, v\rangle + a\langle u_a, v\rangle = 0. \quad (6)$$

Thus (5) and (6) imply

$$F_a(u+v) = F(u_a) + \|Av\|^2 + a\|v\|^2 \geq F_a(u_a),$$

and $F(u_a + v) = F(u_a)$ if and only if $v = 0$, so u_a is the unique minimier of $F_a(u)$.

Theorem 1 is proved.

Theorem 2. If x_0 is the unique minimal-norm solution of (1) and u_a is the unique minimier of $F_a(u)$ then

$$\lim_{a \to 0} \|u_a - x_0\| = 0, \text{with } u_a = A^*\left(AA^* + aI_Y\right)^{-1} f. \quad (7)$$

Proof. Write (4) as $A(A^* w_a - x_0) = -a w_a$. Apply A^* , which is possible because $u_a \in D\left(A^*\right)$, we obtain

$$A^* A(u_a - x_0) = -a u_a \quad (8)$$

Multiply (8) by $u_a - x_0$, we obtain

$$\langle A^* A(u_a - x_0), u_a - x_0\rangle = -a\langle u_a, u_a - x_0\rangle,$$

or

$$\|A(u_a - x_0)\|^2 = -a\|u_a\|^2 - \langle u_a, x_0\rangle \quad (9)$$

Since $a > 0$ this implies

$$\|u_a\|^2 \leq \langle u_a, x_0\rangle.$$

So

$$\|u_a\| \leq \|x_0\|, \forall a > 0.$$

Therefore, one may assume (taking a subsequence) that the sequence u_a weakly converges to an element z, we denote it as $u_n := u_{a_n} \overset{w}{\to} z$, as $a_n \to 0$.

It follows from (9) that

$$\lim_{n\to\infty} \|A(u_n - x_0)\| = 0, \text{ i.e. } \lim_{n\to\infty} \|Au_n - f\| = 0.$$

We shall prove that $z = x_0$.

Let γ run through the set such that $\{A^* Ag\}$ is dense in $N^\perp$, where $N = N(A)$. Note that $N(T) = N(A)$, where $T = A^* A$ [3]. Because of the formulas $X = \overline{R(T)} \oplus N(T)$ [3], then $\{g\} = D(T)$ is dense in X, and the set {Tγ} is dense in $N^\perp$.

Multiply the equation $T(u_a - x_0) = -\alpha u_a$ by g and pass to the limit $\alpha \to 0$, we obtain

$$\langle z - x_0, Tg \rangle = 0.$$

We have $x_0 \perp N$. If $z \perp N$, then $z - z_0 \perp N$ and $z - x_0 \perp N^\perp$, so

$$z - x_0 = 0.$$

One may always assume that $z \perp N$ because $Tu_a = T\tilde{u}_a$, where $\tilde{u}_a$ is the orthogonal projection of u_a onto $N^\perp$. Thus, we have $u_n := u_{a_n} \overset{w}{\to} z$, $\|u_n\| \le \|x_0\|$. Thus implies $\lim_{n\to\infty} \|u_n - x_0\| = 0$ [4]. For convenience for the reader, we prove this claim. Since $u_n := u_{a_n} \overset{w}{\to} z$, one gets $\|x_0\| \le \underline{\lim}_{n\to\infty} \|u_n\|$. The inequality $\|u_n\| \le \|x_0\|$ implies $\overline{\lim}_{n\to\infty} \|u_n\| \le \|x_0\|$. Therefore $\lim_{n\to\infty} \|u_n\| = \|x_0\|$. This and the weakly converge $u_n := u_{a_n} \overset{w}{\to} z$ imply strong convergence

$$\|u_n - x_0\|^2 = \|u_n\|^2 + \|x_0\|^2 - 2\mathrm{Re}\langle u_a, x_0\rangle \to 0,$$

as $n \to \infty$.

Theorem 2 is proved.

Theorem 3. If $\|f_d - f\| \le d$, x_0 is the unique minimal-norm solution of (1), and

$$F_{a,d}(u) = \|Au - f_d\|^2 + a\|u\|^2 \to \min, \tag{10}$$

then there exists a unique global minimier $u_{a,d}$ to (10) and $\lim_{d \to 0} \|u_d - x_0\| = 0$, where $u_d := u_{a(d),d}$ and $a(d)$ is properly chosen, in particular $\lim_{d \to 0} a(d) = 0$.

Proof. The existence and uniqueness of the minimizer $u_{a,d}$ of $F_{a,d}$ follows from Theorem 1 and $u_{a,d} = A^*(Q + aI_Y)^{-1} f_d$. We have

$$\|u_{a,d} - x_0\| \le \|u_{a,d} - u_a\| + \|u_a - x_0\|.$$

By Theorem 2, $\|u_a - x_0\| := h(a) \to 0$, as $a \to 0$, with

$$u_a = A^*(AA^* + aI_Y)^{-1} f = A^*(Q + aI_Y)^{-1} f$$

Let us estimate

$$\|u_{a,d} - u_a\| = \|A^*(Q + aI_Y)^{-1}(f_d - f)\| \le d\|A^*(Q + aI_Y)^{-1}\|$$

By the polar decomposition theorem [5, 6], one has $A^* = UQ^{1/2}$, where U is a partial isometry, so $\|U\| \le 1$. One has,

$$\|A^*(Q + aI_Y)^{-1}\| = \|UQ^{1/2}(Q + aI_Y)^{-1}\|$$

$$\le \|Q^{1/2}(Q + aI_Y)^{-1}\| = \max_{l \ge 0} \frac{l^{1/2}}{l + a} = \frac{1}{2\sqrt{a}},$$

where the spectral representation for Q was used.

Thus,

$$\|u_{a,d} - x_0\| \le \|u_{a,d} - u_a\| + \|u_a - x_0\| \le \frac{d}{2\sqrt{a}} + h(a) \tag{11}$$

For a fixed small $d > 0$, choose α=α(δ) which minimizes the right side of (11), then

$$\lim_{d \to 0} a(d) = 0 \text{ and. } \lim_{d \to 0} \left(\frac{d}{2\sqrt{a(d)}} + h\left(a(d)\right) \right) = 0$$

Theorem 3 is proved.

Remark 1. From (11), we can also choose $a(d) = cd^k$, with any $0 < k < 2$ and c= const > 0. The constant c can be arbitrary.

We can also choose $a(d)$ by a descrepancy principle. For example, consider the equation for finding $a(d)$:

$$\|Au_{a,d} - f_d\| = cd, c = \text{const} > 1. \tag{12}$$

We assume that $\|f_d\| > cd$. That is the content of the following theorem.

Theorem 4. The equation

$$\|Au_{a,d} - f_d\| = cd, c = \text{const} > 1, \|f_d\| > cd, \tag{13}$$

has a unique solution $a = a(d) > 0$, $\lim_{d \to 0} a(d) = 0$, and if $u_d := u_{a(d),d}$, then

$$\lim_{d \to 0} \|u_d - x_0\| = 0.$$

Proof. Let us prove that Eq. (13) has a unique root $a(d) > 0$, $\lim_{d \to 0} a(d) = 0$. Indeed, using the spectial theorem [5, 6], one gets

$$\left\|Au_{a,d} - f_d\right\|^2 = \left\|AA^*(Q + aI_Y)^{-1} f_d\right\|^2$$

$$= \int_0^{\infty} \left|\frac{s}{s+a} - 1\right|^2 d(E_s, f_d, f_d) = a^2 \int_0^{\infty} \frac{d(E_s, f_d, f_d)}{(s+a)^2} =: g(a,d),$$

where E_s is the resolution of the identity of Q.

One has $g(\infty, d) = \left\|f_d\right\|^2 > c^2 d^2$, and $g(0^+, d) = \left\|P_{N^*} f_d\right\|^2$, where P_{N^*} is the orthoprojector onto the subspace $N^* = N(Q) = N(A^*) = R(A)^{\perp}$. Since $f \in R(A)$ and $\left\|f_d - f\right\| \le d$, it follows that $\left\|P_{N^*} f_d\right\| \le d$, so $g(0^+, d) \le d^2$. The function $g(a,d)$ for a fixed $d > 0$ is a continuous strictly increasing function of a on [0, ∞). Therefore, there exists a unique $a(d) > 0$, which solves Eq. (13) if $\left\|f_d\right\| > cd$ and c > 1. Clearly $\lim_{d \to 0} a(d) = 0$, because $\lim_{d \to 0} ca(d) = 0$ and the relation $\lim_{d \to 0} a^2(d) \int_0^{\infty} \frac{d(E_s, f_d, f_d)}{(s + a(d))^2} = 0$ implies $\lim_{d \to 0} a(d) = 0$. The function $\alpha = \alpha(\delta)$ is a monotonically growing function of δ with $\alpha(0^+) = 0$. Let us prove that $\lim_{d \to 0} \left\|u_d - x_0\right\| = 0$, where $u_d := u_{a(d),d}$, and $\alpha(\delta)$ solves Eq. (13). By the definition of u_d, we get

$$\left\|Au_d - f_d\right\|^2 + a(d)\left\|u_d\right\|^2 \le \left\|Ax_0 - f_d\right\|^2 + a(d)\left\|x_0\right\|^2 \le d^2 + a(d)\left\|x_0\right\|^2.$$

Since $\left\|Au_d - f_d\right\|^2 = c^2 d^2 > d^2$, it follows that $\left\|u_d\right\| \le \left\|x_0\right\|$. Thus $u_d \rightharpoonup z$, and as in in the proof of Theorem 2, we obtain $z = x_0$ and $\lim_{d \to 0} \left\|u_d - x_0\right\| = 0$.

Theorem 4 is proved.

Remark 2. Theorems 1-4 are well known in the case of a bounded linear operator A.

If A is bounded linear, then a necessary condition for the minimum of the functional $F_a(u) = \left\|Au - f\right\|^2 + a\left\|u\right\|^2$ is the equation

$$A^*Au + \alpha u = A^* f . \tag{14}$$

Hence in this case conditions are required $f \in D(A^*)$. then, the minimum of the functional

$$F_\alpha(u) = \|Au - f\|^2 + \alpha \|u\|^2$$

is

$$u_\alpha = \left(A^*A + \alpha I_X\right)^{-1} A^* f .$$

If A is a linear, closed, densely defined unbounded operator, by above regularization method then f does not necessarily belong to $D\left(A^*\right)$. Therefore, some changes in the usual theory are necessary. The changes are given in this paper. We prove, among other things, that for any $f \in Y$, in particular for $f \notin D\left(A^*\right)$, the element $u_\alpha := A^*(AA^* + \alpha I_Y)^{-1} f$ is well defined for any $\alpha = \text{const} > 0$, provided that A is a closed, linear, densely difined operator in Hilbert space (Theorem 1).

According to [12], we can replace x_0 is unique minimal-norm solution of (1) by y is solution of (1) satisfying condition $y \perp N(A)$, then theorems 1-4 are still true.

APPLICATIONS

As a simple concrete example of this type of approximation, consider differentiation in

$$X = L^2[0,1].$$

We define the operator $A: D(A) \subset X \to X$ as follows

$$Ax = \frac{dx}{dt}, x \in D(A),$$

with D(A) = {x∈X: x(t) is absolutely continuous on [0,1] and $\frac{dx}{dt} \in X$}. Then D(A) is dense in X since it contains the complete orthonormal set $\{\sin n\pi t\}_{n=1}^{\infty}$. Clearly, A is a linear operator. We show that A is a closed operator in Hilbert space X. Indeed, for suppose $\{x_n\} \subset D(A)$ and

$x_n \to x$ and $Ax_n = x_n' = \frac{dx_n}{dt} \to g$, in each case the convergence being in the $L^2[0,1]$ norm. Since

$$x_n(t) = x_n(0) + \int_0^t x_n'(x)dx,$$

we see that the sequence of constant functions $\{x_n(0)\}$ converges in $L^2[0,1]$ and hence the numerical sequence $\{x_n(0)\}$ converges to some real number C. Now define h ∈ D(A) by $h(t) = C + \int_0^t g(x)dx$. Then, for any t ∈ [0,1], we have by of the Cauchy - Schwarz inequality

$$\begin{aligned} |x_n(t) - h(t)| &= \left| x_n(0) - C + \int_0^t [x_n'(x) - g(x)]dx \right| \\ &\le |x_n(0) - C| + \int_0^t |x_n'(x) - g(x)|dx \\ &\le |x_n(0) - C| + \|x_n' - g\|, \end{aligned}$$

and hence $x_n \to h$ uniformly. Therefore, $x = h \in D(A)$ and $Ax = x' = h' = g$, verifying that the operator A is closed, linear, densely defined in $L^2[0,1]$.

Let

$$D^* = \{g \in D(A) : g(0) = g(1)\}.$$

Then for $x \in D(A)$ and $g \in D^*$, we have

$$\begin{aligned} \langle Ax, g \rangle &= \int_0^1 x'(t)g(t)dt = x(t)g(t)\Big|_0^1 - \int_0^1 x(t)g'(t)dt \\ &= \int_0^1 x(t)g'(t)dt = \langle x, -g' \rangle. \end{aligned}$$

Therefore $D^* \neq D(A^*)$ and $A^* g = -g$, for all $g \in D^*$. On the other hand, if $g \in D(A^*)$, let $g^* = A^* g$. Then

$$\langle Ax, g \rangle = \langle x, g^* \rangle,$$

for all $x \in D(A)$. In particular, for x ≡ 1, we find that $\int_0^1 g^*(t)dt = 0$.

Now let

$$h(t) = -\int_0^t g^*(s)ds.$$

Then $h \in D^*$ and $A^* h = g^* = A^* g$ and hence $h - g \in N(A^*)$. Therefore, $\langle Ax, h - g \rangle = 0$, for all x ∈ D(A). But R(A) contains all continuous function and hence $g = h \in D^*$.

We conclude that

$$D(A^*) = D^* \text{ and } A^* g = -g.$$

If the equation:

$$Ax = f \tag{15}$$

is solvable, then (15) has the unique minimal-normal solution x_0. According to Theorem 1, for any $f \in X = L^2[0,1]$, the problem

$$F_a(u) = \|Au - f\|^2 + a\|u\|^2$$

has a unique solution $u_a = A^*\left(AA^* + aI_X\right)^{-1} f$, where I_X is the identity operator on $X = L^2[0,1]$ for all $f \in X = L^2[0,1]$ does not necessarily belong to $D(A^*)$.

It follows from Theorem 2, then

$$\lim_{a \to 0} \|u_a - x_0\| = 0, \text{ with } u_a = A^*(AA^* + aI_X)^{-1} f$$

It follows from Theorem 3, that if $\|f_d - f\| \le d$ and

$$F_{a,d}(u) = \|Au - f_d\|^2 + a\|u\|^2 \to \min, \tag{16}$$

then there exists a unique global minimier $u_{a,d}$ to (16) and $\lim_{d \to 0} \|u_d - x_0\| = 0$, where $u_d := u_{a(d),d}$ and α(δ) is properly chosen, in particular $\lim_{d \to 0} a(d) = 0$.

It follows from Theorem 4, that the equation

$$\|Au_{a,d} - f_d\| = cd, c = const > 1, \|f_d\| > cd,$$

has a unique solution α = α(δ) > 0, $\lim_{d \to 0} a(d) = 0$, and if $u_d := u_{a(d),d}$, then

$$\lim_{d \to 0} \|u_d - x_0\| = 0.$$

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