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# A Study on the Eigenmodes of the Rectangular Cavity

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## ABSTRACT

**In this study, finite Element method has been carried out to investigate the eigen properties of the rectangular cavity. Because the horizontal length is much larger than that of vertical one, eigenmodes take on various forms in the horizontal direction. The physical properties of rectangular cavities have been considered basing on the higher-order eigenmodes including the ground state. The eigenmodes have been divided into two classes: transverse electric modes and transverse magnetic modes. These classes were depended on the boundary conditionimposed on the cavity surface. The eigenmodes have been described with the electric field, electric potential and electric equipotential contour. The eigenmode was determined by comparing and compensating the electric fieldand potential each other. These results have been revealed by schematic representations.**

**Keyword:** rectangular cavity, eigenmode, transverse electric mode, transverse magnetic mode, boundarycondition.

## INTRODUCTION

Previously, the physical properties of various waveguides made of photonic crystals were analyzed using FEM (Finite Element Method) techniques [1][2]. FEM for photonic crystals made of different materials could accurately derived quantitative results without any contradiction by appropriately setting the mesh elements at their interfaces. The physical quantity to be considered in such a system has been typically obtained from the characteristics of the eigenmode. The pair of eigenmode considered in those studies consists of the electric field and potential resulted from FEM simultaneously. The characteristics of the eigenmode in the waveguide of photonic crystal were determined by the spatial arrangement of the dielectrics in the basis space. In addition, the eigen properties of waveguides exhibiting various geometric structures were also studied [3]. Unlike optical fibers, the waveguides considered there were simple materials with a homogeneous internal structure, but had various geometric structures. Such waveguides were developed to take advantage of the property that the ground state eigenmode resonates with externally injected electromagnetic waves and can propagate over long distances with low energy loss. The eigenmodes were mainly determined through the spatial distribution of the electric field and the potential characteristics were excluded in those studies. If information about the spatial distribution of electric potentials had been included, the physical properties that the eigenmodes were intended to represent could have been revealed more precisely. In other words, by considering the characteristics of the electric potential distributed in space along with the electric field, better physical properties could be expected to be understood.

From those studies, it was found that the resulting eigenmodes were closely related to the material property and the geometric structure of the waveguide. In particular, apart from the diversity of constituent materials, structural properties can be easily changed and the resulting physical characteristics can be broadly understood. Previously, the eigenmodes formed in waveguides of various structures including the circular, square and rectangular etc., have been discussed [4]. However, the cross section of the waveguide can be treated as a two-dimensional cavity. In this study, the eigenmodes are examined formed in an ordinary two-dimensional rectangular-structured cavity. The width relative to height is significantly larger than previous studies on waveguides. The electric potential together with the electric equipotential contour are added to the electric field. The eigenpair is composited with the vector electric field and scalar electric potential. These components complement each other and may further precisely reveal the eigenmode characteristics of the cavity. The eigenmode is obtained through the calculating process of FEM. FEM is based on the governing vector Helmholtz equation. The equation is divided into xy- coordinate system and z-component through variable separation. In this study, the process of establishing matrix eigen-equations based on the Helmholtz equation is not mentioned. This is because the mathematical process was mentioned in detail in the previous study. The figures related to the theory can also be referenced in previous manuscripts [5]. Therefore, the description format of this manuscript deviates from the existing order and is described as follows. This manuscript describes the structure and FEM process, spectrum properties of the rectangular cavity, the result and discussion of the simulation, and conclusion in order.

## THE CAVITY STRUCTURE AND FEM

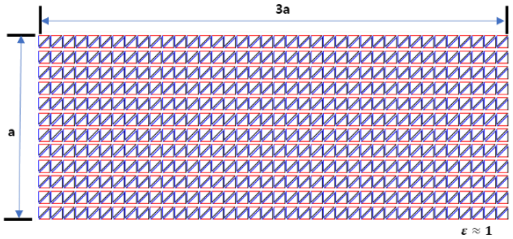



**Figure 1: Rectangular cavity**

The rectangular cavity used in this study is shown in figure 1. As seen in the figure, the horizontal length $L_h$ of the cavity is much larger than vertical one $L_w$. The ratio between them is $L_h : L_w = 3 : 1$. The material in the cavity is homogeneous and it is assumed to be air (the relative dielectric constant $\varepsilon \approx 1$). The cavity is only a partof the waveguide. The vertical cross-sectional area of an infinitely expanded waveguide formed in three- dimensional space become to be a two-dimensional cavity. In this study, the longitudinal direction of the waveguide is set to z-axis and its cross section is considered to be the xy-plane. Since the horizontal length in thex-direction of the cavity is three times longer than the vertical one in the y-direction, the characteristics of a two-dimensional waveguide are expected to appear in some horizontal space.

FEM is performed to find the eigenmodes formed in the rectangular cavity. The internal space of the cavity is decomposed into the mesh of triangular elements. The edges and nodes of the triangular element are pair components that describe the electric field vectors and electric

potentials of the formed eigenmode respectively. The size of the matrix equation that describes the eigenmode of the cavity is generally proportional to the number of edges and nodes forming the triangular element. In this study, the number of edges is 1418 and the number of nodes is 507, so a very large matrix equation of 1925 × 1925 is resulted. In order to solve this equation using a PC (Personal Computer), a sophisticate algorithm must be used because the capacity of PC is limited [7]. So, the original equation is compressed into a low-dimensional matrix using the Arnoldi algorithm. And the Krylov-Schur iteration method is applied on it. In this way, the few reliable eigenmodes are resulted. Those results represent the characteristics of a rectangular cavity very well.

Since the electric field is described as a vector, the edges of each triangular element must be constructed consistently. The common edges of the two triangular elements of the mesh structure must consistently point in the same direction. These characteristics are well depicted in figure 2 in reference [5]. The edges inside the cavity are shared by two triangular elements. The edges across the cavity surface are contained in only one triangular element. The electric field is drawn at the barycentric point of the triangular element. The electric field can be obtained using the edge vectors of the triangular element as follows:

$$\vec{F}t \;=\; \sum_{m=1}^{m=3} \overrightarrow{W}_{mt} F_{mt} \;(where\ m\text{: }the\ edges\ number) \tag{1}$$

$\overrightarrow{W}_{mt}$: $the\ tangential\ edge\ vector\ component\ m$

$F_{mt}$ : $the\ magnitude\ of\ the\ electric\ field\ component\ m$

The node points of the triangular element describe the electric potential which is simultaneously obtained with the electric field. The node point of the triangular element describes the electric potential. These values are also used when drawing equipotential contours by interpolation. Except for the nodes on the surface, those in the inside of the cavity are shared with six neighboring triangular elements. A node present on the surface of the cavity is shared with three neighboring triangular elements. And the node exist at the conner of the cavity are involved only one elementary triangle. The electric potential at a point inside the triangular element can be obtained by combining the shape function as follows:

$$F_z \;=\; \sum_{m=1}^{m=3} N_n F_{nz} \;(where\ n\text{: }the\ node\ number) \tag{2}$$

$N_n$: $the\ shape\ function\ of\ the\ node\ component\ n$

$F_{nz}$: $the\ magnitude\ of\ the\ electric\ potential\ component\ n$

The values $F_{mt}$ and $F_{nz}$ are obtained as a result of the FEM process [6]. These values are the column matrix components of the similarity-transformation matrix that transforms the eigen matrix into the Schur matrix. Among the column matrix components, the values resolving to the edges of the triangular elements are substituted into equation (1) to represent the electric field vector at the barycentric point. And among the column matrix components, the values resolving

to the nodes of the triangular elements are substituted into equation (2) to represent the electric potential at the arbitrary point of that element.

The surface of the cavity is the boundary that separates the interior from the area surrounding it. Even if the dielectric constant inside the cavity has a value similar to that of the outside surrounding it, an eigenmode can be uniquely formed by this boundary. In simulation such as FEM, the distinction between TE-mode (Transverse Electric mode) and TM-mode (Transverse Electric mode) is determined by the boundary conditions of the surface. The TE-mode uses all matrix components to obtain eigen pairs, but the TM-mode is obtained by excluding edge and nodal components present on the cavity surface. Therefore, the matrix component of TM-mode can be calculated more easily because the edge and node components present on the surface are excluded.

## THE EIGENMODE OF THE RECTANGULAR CAVITY

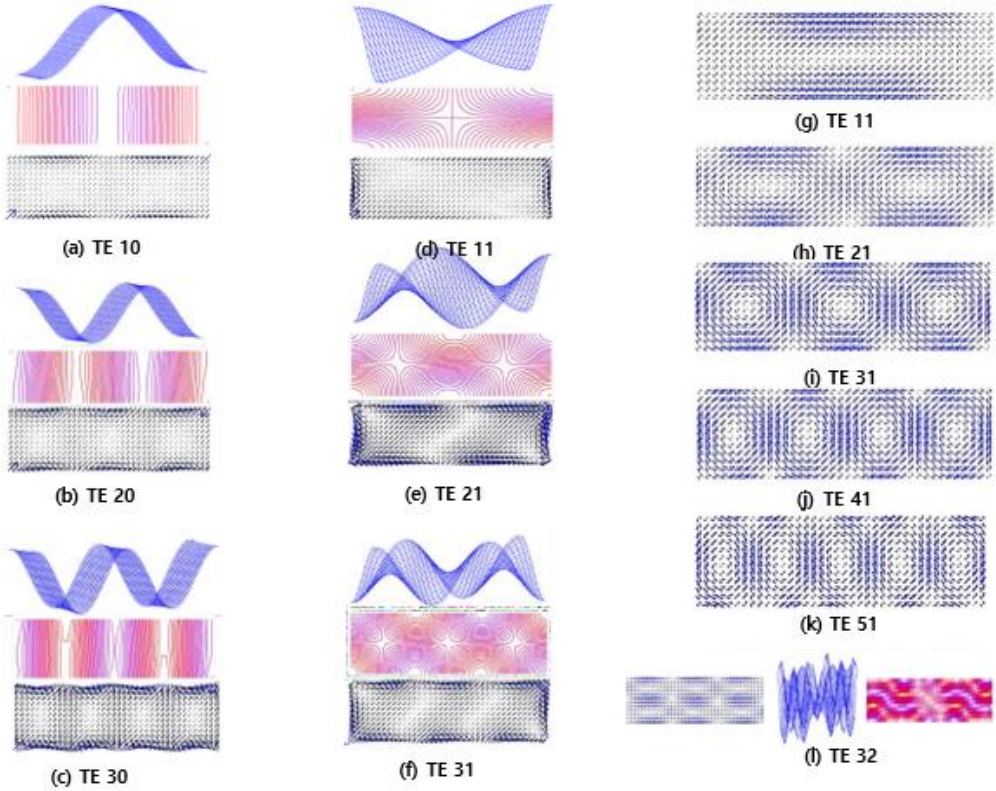



**Figure 2: Schematic representation of TE-mode**

Figure 2. show the schematic representation of the eigen TE-modes. From (a) to (f), these are described as three-dimensional potentials, equipotential contours, and electric fields. From (g) to (l) are similar to the spectra mentioned in the previous manuscript [4], and their characteristics were explained only by the electric field. The lowest order eigenmode is the spectrum in Figure 2(g) which is mentioned in the reference [4]. There are varioustypes of the eigenmodes appearing in the rectangular cavity. The figure 2 of (a)~(c), (d)~(f) and(g)~(l) are eigenmode groups of the various type. The eigenvalues corresponding to each spectrum cannot

be used as a directcomparison standard because the horizontal length was finely adjusted to improve the resolution of the eigenmode for convenience during the calculation process. However, it is clear that the ground state corresponding to each eigenmode group takes the form (a), (d) and (g). The eigenmodes of each group show a consistent development as the eigen-order increases. Considering the spectrum (a) to (c), it shows that as the order of eigen number increases, the wave number ($k = 2\pi / \lambda\ [1/metre], \lambda = 2 \times 3a[metre]$) inside the cavity promote sequentially. As can be seen from the electric potential eigenmodes, in the horizontal direction, the wave number increases as $(a) \rightarrow k_x = 1$, $(b) \rightarrow k_x = 1.5$ $and$ $(c) \rightarrow k_x = 2$ as the order of the eigenmode increases. The wavelength corresponding to each wave number is $(a) \rightarrow \lambda_x = 3a$, $(b) \rightarrow \lambda_x = 2a$ $and$ $(c) \rightarrow \lambda_x = 1.5a$ respectively. Equipotential contours, which are projections of potentials on a two-dimensional plane, describe these propertiesvery well. In addition, the eigenmode expressed as an electric field well represents the standing wave characteristics formed in a closed system rectangular cavity. Since wave characteristics do not appear in the y- direction, it can be asserted that resonance occurs only in the horizontal direction. Their eigenmodes are determined as $(a) \rightarrow$ TE 10, $(b) \rightarrow$ TE 20 $and$ $(c) \rightarrow$ TE 30 respectively. There is another class of the eigenmodes different from (a)~(c). For these, eigenmode components along to the y-direction clearly exist. As can be seen in the spectra (d)~(f), eigenmode components are formed in the x and y-directions. The spectra showthe integer wavelengths developed in the x, y-direction. The eigenmodes represented by these spectra are as TE 11, TE 21 and TE 31 respectively. The wavenumbers and wavelengths in x-direction are $(d) \rightarrow k_x = 1$, $(e) \rightarrow k_x = 2$ $and$ $(f) \rightarrow k_x = 3$ and $(d) \rightarrow \lambda_x = 3a$, $(e) \rightarrow \lambda_x = 1.5a$ $and$ $(f) \rightarrow \lambda_x = 1a$ respectively. The wavenumbers and wavelengths in y-direction are $(d) \rightarrow k_y = 1$, $(e) \rightarrow k_y = 1$ $and$ $(f) \rightarrow k_y = 1$ and $(d) \rightarrow \lambda_y = a$, $(e) \rightarrow \lambda_y = a$ $and$ $(f) \rightarrow \lambda_y = a$ respectively. Spectra (g)~(l) are similar to those mentioned in reference [4]. The rectangular cavity treated in this study has a larger horizontal length at a ratio of 3:2. As the length in the horizontal direction increases, deformation of the eigenmode is expected to occur. When compared superficially with that of reference [4], the eigenmodes of this manuscript consistently show an increase in the mode component in the x-direction excluding only spectrum (l). There is no problem in determining the eigenmode using the spectra described by the component of electric field. However, when considering the component of electric potential, much confusion must be taken into account. Like the spectrum (l), it is difficult to find the electric potential corresponding to the electric field in each spectrum. Even if it exists, it is not as orderly and consistent as the spectrum that describes the electric field. It is difficult to determine the eigenmode with the electric potential for each spectrum. Therefore, it is satisfied in determining the eigenmodes of the spectra solely through the characteristics of the electric field. Their eigenmodes are determined as (g)→TE 11, (h)→TE 21, (i)→TE 31, (j)→TE 41, (k)→TE 51 and (l)→TE 32 respectively. This method is the same as reference [4],

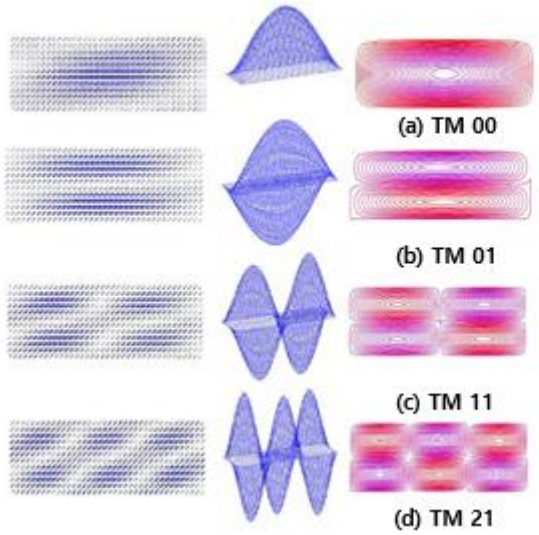



**Figure: 3_1 TM-mode**

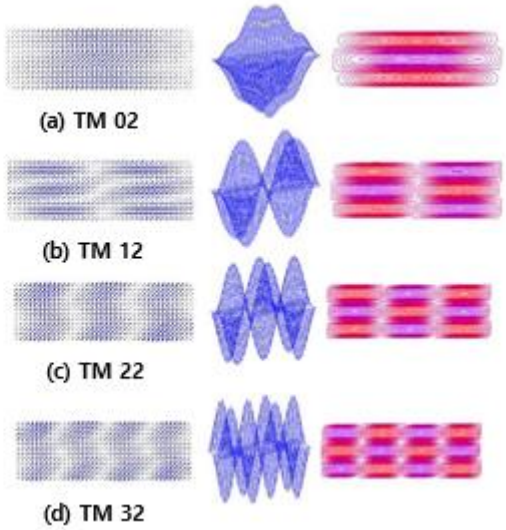



**Figure: 3_2 TM-mod**

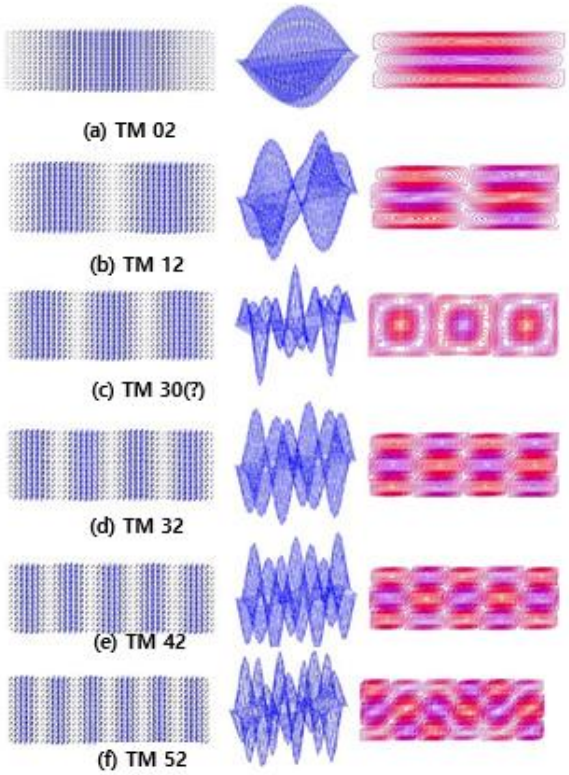



**Figure: 3_3 TM-mode**

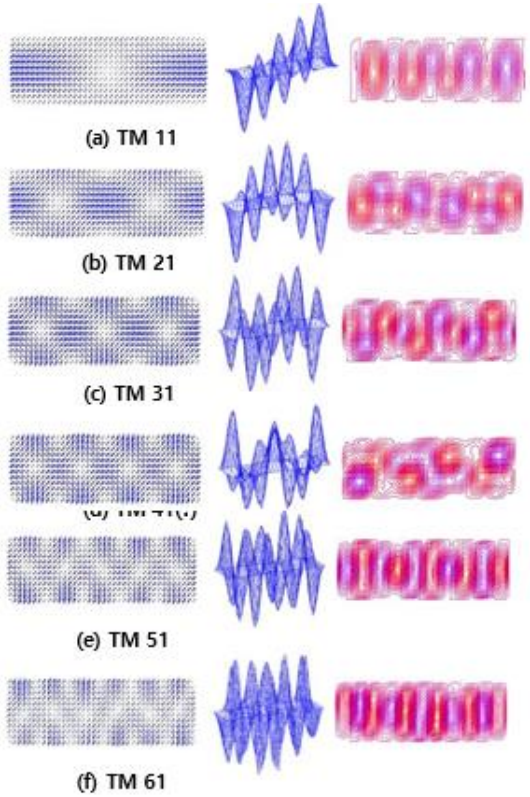



**Figure: 3_4 TM-mode**

which obtained the eigenmode using only the electric field without calculating the potential. Due to the boundary conditions set on the cavity surface, TM-mode imposes several constraints on the electric field and potential. First, only the component of the electric field perpendicular to the surface can be distributed. And secondly, the electric equipotential line only takes the form of a tangent to the surface. Figure 3_1, 3_2, 3_3 and 3_4, represent eigenmodes that

satisfies all these characteristics. These are sequentially organized by grouping together complex but similar eigenmodes. Figure 3_1 is the lowest-order TM eigenmode spectra represented by the electric field, 3_Dim potential, and equipotential contour. These have clear outlines representing the x-mode and y-mode, so characteristics of the eigenmode can be determined without any problems. The eigenmodes for each of the figure 3_1 spectrum is determined as $(a) \rightarrow$ TM 00, $(b) \rightarrow$ TM 01, $(c) \rightarrow$ TM 11, $and$ $(d) \rightarrow$ TM 21, respectively. The difference between the spectrum explaining the TE mode in figure 2 and the TM in figure 3_1 is the component of the cavity surface. In the process of FEM calculation, the TM mode excluded the surface edges and nodes of the cavity, but in the TE mode, those components remained. In the case of figure 2(l), the surface components of the cavity participate in determining the eigenmode, but in the case of TM 00 in spectrum 3_1 (a), these components are excluded. The existed TM 00 mode in the spectrum reveals the lowest eigenstate. Figure 3_2 shows another group of TM modes. The eigenmodes in this figure also show electric fields and potentials with good resolution. Therefore, for each spectrum, reasonable mode characteristics can be given with high reliability. The eigenmodes for each of the figure 3_2 spectra are determined as $(a) \rightarrow$ TM 02, $(b) \rightarrow$ TM 12, $(c) \rightarrow$ TM 22, $and$ $(d) \rightarrow$ TM 32, respectively. Figure 3_3 represents another type of eigenmode with good resolution, except for (c). The spectrum (c) shows the same trend of variation of the electric field as the others, but the electric potential reveals a different type. As can be seen in the figure, the electric potential takes a noticeably different form compared to others. But, even if the shape is different, it is possible to trace the electric field from this electric potential. The horizontal components of the electric potential are combined to produce the vertical lines of the electric field eigenmode. Three main peaks of electric potential exist in the region where the electric field is extremely developed. Small peaks around the main electric potential correspond to weak parts of the electric field. Its means that there is not only one electric potential corresponding to an electric field, but also that of other forms can exist to establish normal eigenmode. From the above discussion, it can be identified that this process important in understanding the physical properties of the main eigenmodes as well as the resonance characteristics of various rectangular cavities. Except for the potential problem in (c), eigenmodes can be imposed on each spectrum through an electric field or electric potential. What is noteworthy is that in all spectra, only the vertical component of the electric field appears. And in the case of the electric potential spectrum, all mode boundaries in the y-direction appear by two lines except for that of (c). Therefor for each spectrum of figure 3_3, an eigenmode can be assigned as follows (a) → TM 02, (b) → TM 12, (c) → TM 22, (d) → TM 32 (e) → TM 42 and (f) → TM 52. Figure 3_4 shows another type of eigenmode group. The electric field eigenmodes belonging to this group develop consistently and exhibit the characteristics of TM modes. The electric field is incident perpendicular to the cavity surface and the electric potential is distributed there only parallelly. As the eigen order increases, the spatial distribution of the electric field develops regularly. But as in figure 3_3 (c), the potential component in figure 3_4 (d) shows an irregular shape, unlike the others. It is difficult to reveal the principles of combination of components that cause irregularities of electric potential. There is currently no clear explanation for this issue and more research is needed. The eigenmode assignment for each spectrum is based on the electric field and uses the electric potential as an auxiliary means. Using the same method previously assigning the mode type to each spectrum, the following results are obtained as follows (a) → TM 11, (b) → TM 21, (c) → TM 31, (d) → TM 41 (e) → TM 51 and (f) → TM 61.

## CONCLUSION

The eigenmodes formed in the rectangular cavity have been obtained by using the FEM technique. According to the boundary condition of the cavity surface, eigen modes have been divided into TE and TM modes. Each eigenmode has been summarized in terms of electric field, 3-Dim potential, and equipotential contours and simply revealed by the schematic representation. There have been some eigenmodes of electric potential components shows an irregular shape, unlike the others. It has been difficult to reveal the principles of combination of components that cause irregularities of the electric potential. There has been currently no clear explanation for this issue and more research would be needed.

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