Transactions on Engineering and Computing Sciences - Vol. 11, No. 3
Publication Date: June 25, 2023
DOI:10.14738/tecs.113.14807.
Feng, C. (2023). Dynamical Behavior for an Extended Boissonade-De Kepper Model. Transactions on Engineering and Computing Sciences, 11(3). 88-97.

SERVICES FOR SCIENCE AND EDUCATION

Dynamical Behavior for an Extended Boissonade-De Kepper Model

Chunhua Feng
Department of Mathematics and Computer Science,
Alabama State University, Montgomery, USA

ABSTRACT

In this paper, an extended Boissonade-De Kepper model with discrete delays is considered. By linearizing the system at the equilibrium point and analyzing the instability of the linearized system, two sufficient conditions to guarantee the existence of periodic solutions of the system are obtained. It is found that under suitable conditions on the parameters, time delay affects the stability of the system. The present method does not need to consider a bifurcating equation. Some numerical simulations are provided to illustrate our theoretical prediction. An interesting problem is solved.

Keywords: extended Boissonade-De Kepper model, delay, instability, periodic solution

INTRODUCTION

In 1980, Boissonade and Kepper provided a mathematical model for an open chemical system, the model is governed by the following equations:

$$\begin{cases} x'(t) = -(x^3 - \mu x(t) + \Lambda) - k\mathrm{y} \\ y'(t) = (x - y)/\tau \end{cases} \tag{1}$$

where k, μ, and Λ are control parameters, and τ is the feedback time. The stability and limit cycle oscillations were discussed [1]. Similar to model (1), many authors have investigated the nonlinear dynamics in a number of chemical systems [2-5]. In model (1), when $x = y$, system (1) is at steady state. By replacing $y(t)$ as $x(t-\tau)$, one can obtain a reduced Boissonade-De Kepper model as follows:

$$x'(t) = -(x^3 - \mu x(t) + \Lambda) - kx(t-\tau) \tag{2}$$

Epstein and Lao derived general results in the linear stability analysis for model (2) [6].

Recently, Abdulrahuman et al. extended model (2) to the following delayed equation:

$$x'(t) = -\eta x^3 + \mu\eta x(t) - \Lambda - kx(t-\tau) \tag{3}$$

The stability, convergence and bifurcation for a reduced Boissonade-De Kepper model (3)

were discussed [7]. Indeed, the reduced Boissonade-De Kepper model is a special case of the following differential equation:

$$x'(t) = a_3x^n + a_2x^2 + a_1x(t) + a_0 + a_4x(t-\tau) \tag{4}$$

where $n \in (2.8,\ 3.2)$. As the authors in [7] pointed out that it would be interesting to investigate the stability and dynamical properties in model (4) when $n \in (2.8,\ 3.2)$. On the other hand, a coupled system of simple oscillators may often arise many interesting phenomena. Specially, the coupled oscillators with time delay have been studied for various physical, chemical, and biological problems [8-15]. For example, Zhang and Gu investigated a coupled van der Pol model with two delays:

$$\begin{cases} x_1'(t) = x_2(t) \\ x_2'(t) = -(1-\alpha)x_1(t) + \alpha y_1(t-\tau_2) - \varepsilon(x_1^2(t)-1)x_2(t) \\ y_1'(t) = y_2(t) \\ y_2'(t) = -(1-\alpha)y_1(t) + \alpha x_1(t-\tau_1) - \varepsilon(y_1^2(t)-1)y_2(t) \end{cases} \tag{5}$$

The authors analyzed the stability and Hopf bifurcation for model (5). The critical delay values and existence conditions for the Hopf bifurcation were obtained [8]. Achouri et al. have discussed the following coupled time delay system [14]:

$$\begin{cases} x_1'(t) = -x_1(t)(x_1(t)\ -\ 1)(x_1(t)\ -\ a) - y_1(t)\ +\ c_1f_1(x_2(t\ -\ \tau_2) \\ y'_1(t) = b_1\big(x_1(t) - \gamma_1y_1(t)\big) \\ x_2'(t) = -x_2(t)(x_2(t)\ -\ 1)(x_2(t)\ -\ a)\ -\ y_2(t)\ +\ c_2f_2(x_1(t\ -\ \tau_1) \\ y'_2(t) = b_2\big(x_2(t) - \gamma_2y_2(t)\big) \end{cases} \tag{6}$$

By using the normal form theory and reduction on the centre manifold, the truncated normal form was obtained, the dynamical behaviours of bifurcation diagram were studied. In this paper, we extend model (1) to the following coupled time delay system:

$$\begin{cases} x_1'(t) = -\eta_{11}x_1^{\alpha}(t) + \eta_{12}x_1^2(t) + \eta_{13}x_1(t) - a_1y_1(t) + k_2x_2(t-\tau_2) + \Lambda_1 \\ y'_1(t) = b_1x_1(t) - c_1y_1(t) \\ x_2'(t) = -\eta_{21}x_2^{\alpha}(t) + \eta_{22}x_2^2(t) + \eta_{23}x_2(t) - a_2y_2(t) + k_1x_1(t-\tau_1) + \Lambda_2 \\ y'_2(t) = b_2x_2(t) - c_2y_2(t) \end{cases} \tag{7}$$

where $\alpha \in (2.8,\ 3.2)$, all control parameters are real constants. By means of the mathematical analysis method, the stability and oscillation of the solutions for system (7) are investigated.

For convenience, let $u_1(t) = x_1\ (t-\tau_1)$, $v_1(t) = y_1\ (t-\tau_1)$, $u_2\ (t) = x_2\ (t)$, $v_2\ (t) = y_2\ (t)$, then system (7) is equivalent to the following system:

$$\begin{cases} u'_1(t) = -\eta_{11}u_1^{\alpha}(t) + \eta_{12}u_1^2(t) + \eta_{13}u_1(t) - a_1v_1(t) + k_2u_2(t-\tau) + \Lambda_1 \\ v'_1(t) = b_1u_1(t) - c_1v_1(t) \\ u'_2(t) = -\eta_{21}u_2^{\alpha}(t) + \eta_{22}u_2^2(t) + \eta_{23}u_2(t) - a_2v_2(t) + k_1u_1(t) + \Lambda_2 \\ v'_2(t) = b_2u_2(t) - c_2v_2(t) \end{cases} \tag{8}$$

where $\tau = \tau_1 + \tau_2$. Assume that $(u_1^*, v_1^*, u_2^*, v_2^*)^T$ is an equilibrium point of system (8), namely,

$$\begin{cases} -\eta_{11}u_1^{*\alpha} + \eta_{12}u_1^{*2} + \eta_{13}u_1^* - a_1v_1^* + k_2u_2^* + \Lambda_1 = 0 \\ b_1u_1^* - c_1v_1^* = 0 \\ -\eta_{21}u_2^{*\alpha} + \eta_{22}u*_2^2 + \eta_{23}u*_2 - a_2v*_2 + k_1u*_1 + \Lambda_2 = 0 \\ b_2u_2^* - c_2v_2^* = 0 \end{cases} \tag{9}$$

By the Taylor expansion formula we have $u_i^{\alpha} = u_i^{*\alpha} + \alpha\ u_i^{*(\alpha-1)}(u_i - u_i^*) + \frac{\alpha(\alpha-1)}{2!}u_i^{*(\alpha-2)}(u_i - u_i^*)^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}u_i^{*(\alpha-3)}(u_i - u_i^*)^3 + \cdots + \frac{\alpha(\alpha-1)\ldots(\alpha-n+1)}{n!}\ \theta_i^{(\alpha-n)}(u_i - u_i^*)^n$, where $u_i = u_i(t)$ and θ_i are two appropriately chosen values between u_i and u_i^* (i= 1, 2). From system (8) and (9) we have:

$$\begin{cases} u'_1(t) = -\eta_{11}\left[\alpha\ u_1^{*(\alpha-1)}(u_1 - u_1^*) + \frac{\alpha(\alpha-1)}{2!}u_1^{*(\alpha-2)}(u_1 - u_1^*)^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}u_1^{*(\alpha-3)}(u_1 - u_1^*)^3 + \cdots\right] \\ +\eta_{12}(u_1^2 - u_i^{*2}) + \eta_{13}(u_1 - u_1^*) - a_1(v_1 - v_1^*) + k_2(u_2(t-\tau) - u_2^*) \\ v'_1(t) = b_1(u_1 - u_1^*) - c_1(v_1 - v_1^*) \\ u'_2(t) = -\eta_{21}\left[\alpha\ u_2^{*(\alpha-1)}(u_2 - u_2^*) + \frac{\alpha(\alpha-1)}{2!}u_2^{*(\alpha-2)}(u_2 - u_2^*)^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}u_2^{*(\alpha-3)}(u_2 - u_2^*)^3 + \cdots\right] \\ +\eta_{22}(u_2^2 - u_2^{*2}) + \eta_{23}(u_2 - u_2^*) - a_2(v_2 - v_2^*) + k_1(u_1 - u_1^*) \\ v'_2(t) = b_2(u_2 - u_2^*) - c_2(v_2 - v_2^*) \end{cases} \tag{10}$$

Noting that $\eta_{12}(u_1^2 - u_1^{*2}) = \eta_{12}(u_1 - u_1^*)(u_1 + u_1^*) = \eta_{12}(u_1 - u_1^*)(u_1 - u_1^* + 2u_1^*)$. Make the change of variables $u_i\ (t) \to u_i\ (t) - u_i^*$, $v_i\ (t) \to v_i\ (\mathrm{t}) - v_i^*$ (i = 1, 2), the linearized system of (10) is the following:

$$\begin{cases} u'_1(t) = r_1u_1(t) - a_1v_1(t) + k_2u_2(t-\tau) \\ v'_1(t) = b_1u_1(t) - c_1v_1(t) \\ u'_2(t) = r_2u_2(t) - a_2v_2(t) + k_1u_1(t) \\ v'_2(t) = b_2u_2(t) - c_2v_2(t) \end{cases} \tag{11}$$

where $r_1 = -\eta_{11}\alpha u_1^{*(\alpha-1)} + 2\eta_{12}u_1^* + \eta_{13}$, $r_2 = -\eta_{21}\alpha u_2^{*(\alpha-1)} + 2\eta_{22}u_2^* + \eta_{23}$. System (11) can be written as a matrix form

$$u'(\mathrm{t}) = Au(t) + Bu(t - \tau) \tag{12}$$

where $u(t) = [u_1(t), v_1(t), u_2(t), v_2(t)]^T$, $u(t-\tau) = [u_1(t-\tau), v_1(t-\tau), u_2(t-\tau), v_2(t-\tau)]^{\mathrm{T}}$, and

$$A = \left(a_{\mathrm{ij}}\right)_{4\times4} = \begin{pmatrix} r_1 & -a_1 & 0 & 0 \\ b_1 & -c_1 & 0 & 0 \\ k_1 & 0 & \mathrm{r}_2 & -a_2 \\ 0 & 0 & b_2 & -\mathrm{c}_2 \end{pmatrix},$$

$$B = \left(b_{\mathrm{ij}}\right)_{4\times4} = \begin{pmatrix} 0 & 0 & k_2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

PRELIMINARIES

From model (7) we have the following lemmas:

Lemma 1 Suppose that for the parameters η_{11}, η_{21}, c_1, c_2, we have $\eta_{11} > 0$, $\eta_{21} > 0$, $c_i > 0$ ($i = 1, 2$). Then all solutions of system (7) are bounded.

Proof To show the boundedness of the solutions of system (7), we construct a Lyapunov function V (t) = $\Sigma_{\mathrm{i}=1}^{2}\ \frac{1}{2}(x_{\mathrm{i}}^2(t) + y_{\mathrm{i}}^2(t))$. Calculating the derivative of V (t) through system (7) we have

$$\begin{gathered} \mathrm{V}'(\mathrm{t})|_{(7)} = \sum_{\mathrm{i}=1}^{2}\left(x_{\mathrm{i}}(t)x'_{\mathrm{i}}(t) + y_{\mathrm{i}}(t)y'_{\mathrm{i}}(t)\right) \\ = -\eta_{11}x_1^{\alpha+1} + \eta_{12}x_1^3 + \eta_{13}x_1^2 - \alpha_1x_1y_1 + k_2x_1x_2 + \Lambda_1x_1 + b_1x_1y_1 - c_1y_1^2 \\ -\eta_{21}x_2^{\alpha+1} + \eta_{22}x_2^3 + \eta_{23}x_2^2 - \alpha_2x_2y_2 + k_1x_1x_2 + \Lambda_2x_2 + b_2x_2y_2 - c_2y_2^2 \end{gathered} \tag{13}$$

Noting that as $x_{\mathrm{i}}(t)$ tend to positive infinity, $x_{\mathrm{i}}^{\alpha+1}$are higher order infinity than x_i^3 , x_i^2, y_i^2 , x_1x_2, x_iy_i, x_2y_2 ($i = 1, 2$). Since $\eta_{11} > 0$, $\eta_{21} > 0$, $c_i > 0$ ($i = 1, 2$), hence there exists suitably large $N > 0$ such that $\mathrm{V}'(\mathrm{t})|_{(7)} < 0$ as $x_{\mathrm{i}}(t) > N$, $y_{\mathrm{i}}(t) > N (i = 1, 2)$. This means that all solutions of system (7) are bounded.

Lemma 2 Suppose that matrix $C = (A + B)$ is a nonsingular matrix, then the system (7) has a unique equilibrium point.

Proof Since matrix C (= A + B) is a nonsingular matrix, then system (11) has a unique trivial solution according to the basic linear algebraic theory. This means that system (10) has a unique equilibrium point $(u_1^*, v_1^*, u_2^*, v_2^*)^T$, implying that system (7) has a unique equilibrium point. It is known that the zero equilibrium point of the system (11) (or (12)) corresponds the equilibrium point $(u_1^*, v_1^*, u_2^*, v_2^*)^T$ of the equivalent system (10). In what follows, we only consider the instability of the zero equilibrium point of the system (11).

THE EXISTENCE OF PERIODIC SOLUTION

Since the system (11) is a linearized system of (10). Thus, one can see that the system (10) is a disturbed system of (11). If the trivial solution of the system (11) is unstable, then the trivial solution of the system (10) is also unstable according to the theory of functional differential equations [16]. So, we have the following Theorems.

Theorem 1. Assume that the system (11) has a unique trivial solution, α_1 , α_2 , α_3 , α_4 are characteristic values of matrix A. If there is a characteristic value, say $\alpha_1 > 0$, or $\mathrm{Re}(\alpha_1) > 0$. Then the trivial solution of system (11) (thus the system (10)) is unstable, implying that there exists a limit cycle in the system (7), namely, system (7) has a periodic solution. If $\alpha_i < 0$ or $\mathrm{Re}(\alpha_i) < 0$ ($i = 1, \cdots, 4$), then the trivial solution of system (11) is stable, implying that the equilibrium point of the system (7) is stable.

Proof. We will show that the trivial solution of the system (11) is unstable if $\alpha_1 > 0$, or $\mathrm{Re}(\alpha_1) > 0$. Obviously, all characteristic values of matrix B are zeros. Since α_1, α_2, α_3, α_4 are characteristic values of matrix A, then the characteristic equation of the system (11) is the following:

$$\Pi_{i=1}^{4} \lambda - \alpha_i = 0 \tag{14}$$

If $\alpha_1 > 0$, or $\mathrm{Re}(\alpha_1) > 0$, this means that system (11) has a positive characteristic value or positive real part characteristic value. It suggests that the trivial solution of the system (11) (also the disturbed system (10)) is unstable. Equivalently, the unique equilibrium point of the system (7) is unstable. This instability of the unique equilibrium point together with the boundedness of the solutions will force system (7) to generate a limit cycle, namely, a periodic solution according to the extended Chafee's criterion [see 17] and the appendix of [18]. If $\alpha_i < 0$ or $\mathrm{Re}(\alpha_i) < 0$ ($i = 1, \cdots, 4$), according to the basic differential equation theory, the trivial solution of system (11) is stable. Noting that the nonlinear terms of system (8) are higher order infinitesimal values as $u_i\ (t) \to u_i^*$, $v_i\ (t) \to v_i^*$ (i = 1, 2), implying that the equilibrium point $(u_1^*, v_1^*, u_2^*, v_2^*)^T$ is stable. This means that system (7) has a stable equilibrium point. The proof is completed.

Let $\sigma = \max\{r_1 + |b_1| + |k_1|, -c_1 + |-a_1|, r_2 + |b_2|, -c_2 + |-a_2|\}$, then we have

Theorem 2. Assume that the system (11) has a unique trivial solution. If the following condition holds

$$\sigma + k_2 > 0 \tag{15}$$

Then the trivial solution of the system (11) is unstable, implying that there exists a limit cycle of the system (7), namely, the system (7) has a periodic solution.

Proof. To prove the instability of the trivial solution of the system (11), Let $w(t) = |u_1(t)| + |v_1(t)| + |u_2(t)| + |v_2(t)|$. Therefore, $w(t) > 0$ for $t > 0$, and

$$w'(t) \leq \sigma w(t) + k_2 w(t - \tau) \tag{16}$$

Specifically, consider a scalar equation

$$v'(t) = \sigma v(t) + k_2 v(t - \tau) \tag{17}$$

According to the comparison theory of differential equation we have $w(t) \leq v(t)$. We claim that the trivial solution of the equation (17) is unstable. Indeed, the characteristic equation of (17) is the follows:

$$\lambda = \sigma + k_2 e^{-\lambda\tau} \tag{18}$$

Consider a function $\varphi(\lambda) = \lambda - \sigma - k_2 e^{-\lambda\tau}$. Then $\varphi(\lambda)$ is a continuous function of λ. Noting that $\varphi(0) = -\sigma - k_2 = -(\sigma - k_2) < 0$. Obviously, there exists a $L > 0$ such that $\varphi(L) = L - \sigma - k_2 e^{-L\tau} > 0$. By the Intermediate Value Theorem, there exists a $\lambda_0 \in (0, L)$ such that $\varphi(\lambda_0) = 0$. In other words, there exists a positive characteristic root of the equation (18), this means that the trivial solution of system (17) is unstable, implying that the trivial solution of system (16), thus (11) is unstable. Similar to Theorem 1, system (7) has a periodic solution. The proof is completed.

COMPUTER SIMULATION RESULT

In model (7), we select the parameters as follows: $\alpha = 2.85$, $\eta_{11} = 1.15$, $\eta_{12} = 0.5$, $\eta_{13} = 0.65$, $\eta_{21} = 1.25$, $\eta_{22} = 0.4$, $\eta_{23} = 0.75$, $a_1 = 0.8$, $a_2 = 0.76$, $k_1 = 1.25$, $k_2 = 1.32$, $b_1 = 1.75$, $b_2 = 1.45$, $c_1 = 1.55$, $c_2 = 1.65$, $\Lambda_1 = 2.5$, $\Lambda_2 = 3.5$, then the unique positive equilibrium point $(x_1^*, y_1^*, x_2^*, y_2^*)^T = (1.6665, 1.8813, 1.7105, 1.5042)^T$. Thus, $r_1 = -6.1146$, $r_2 = -7.4894$. The characteristic values of matrix A are -7.3051, -5.7898, -1.8802, -1.8449, the solution is convergent (see Fig.1).

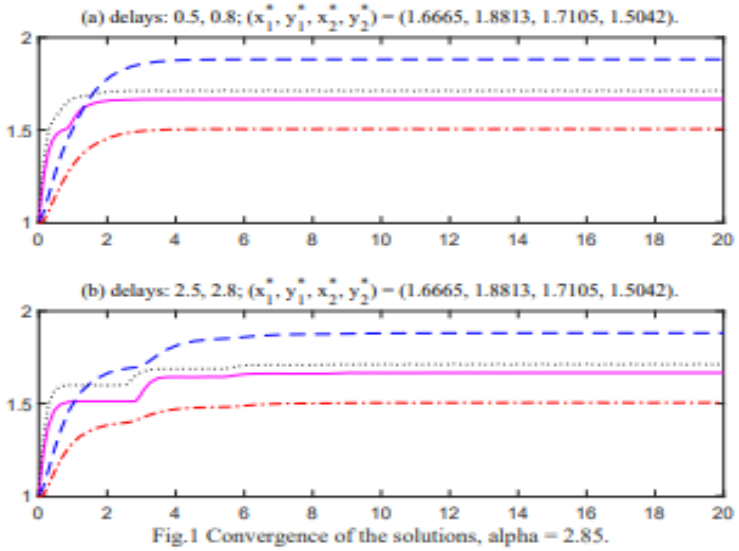


Fig.1 Convergence of the solutions, alpha = 2.85.

When we select α = 3.15, all parameters are the same as figure 1, then $(x_1^*, y_1^*, x_2^*, y_2^*)^T = (1.5503, 1.7503, 1.5936, 1.4023)^T$, and $r_1 = -7.0944$, $r_2 = -8.7013$. The characteristic values of matrix A are −8.5414, −6.8292, −1.8099, −1.8152, the solution also is convergent (see Fig.2).

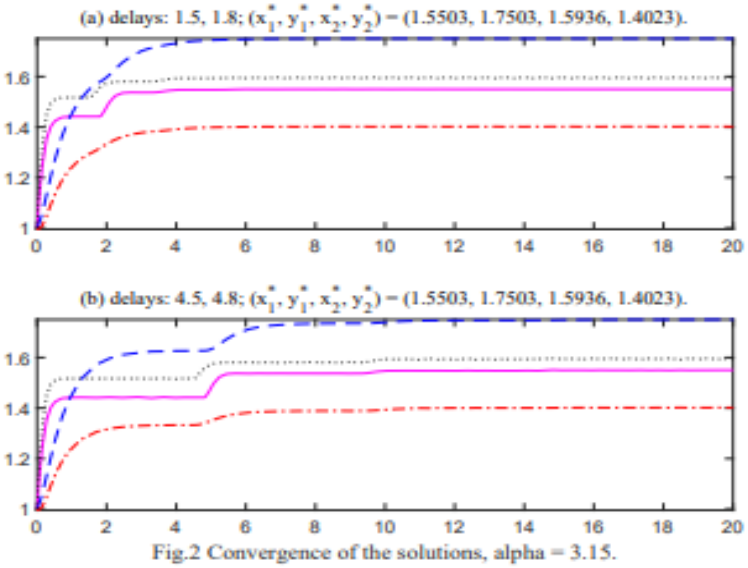


Fig.2 Convergence of the solutions, alpha = 3.15.

When we select α = 2.90, $\eta_{11} = 0.28$, $\eta_{12} = 0.15$, $\eta_{13} = 0.25$, $\eta_{21} = 0.25$, $\eta_{22} = 0.14$, $\eta_{23} = 0.28$, $a_1 = 0.95$, $a_2 = 0.76$, $k_1 = -1.96$, $k_2 = -1.92$, $b_1 = 0.75$, $b_2 = 0.85$, $c_1 = 0.55$, $c_2 = 0.65$, $\Lambda_1 = 0.5$, $\Lambda_2 = 0.8$, then the characteristic values of matrix A are 0.3154, −0.1224 ± 0.2812i, −0.0496, we see that 0.3154 > 0, the condition of Theorem 1 is satisfied, there exists a periodic oscillatory solution around the equilibrium point $(x_1^*, y_1^*, x_2^*, y_2^*)^T = (0.0156, 0.2820, -0.0474, -0.4378)^T$ (see Fig. 3).

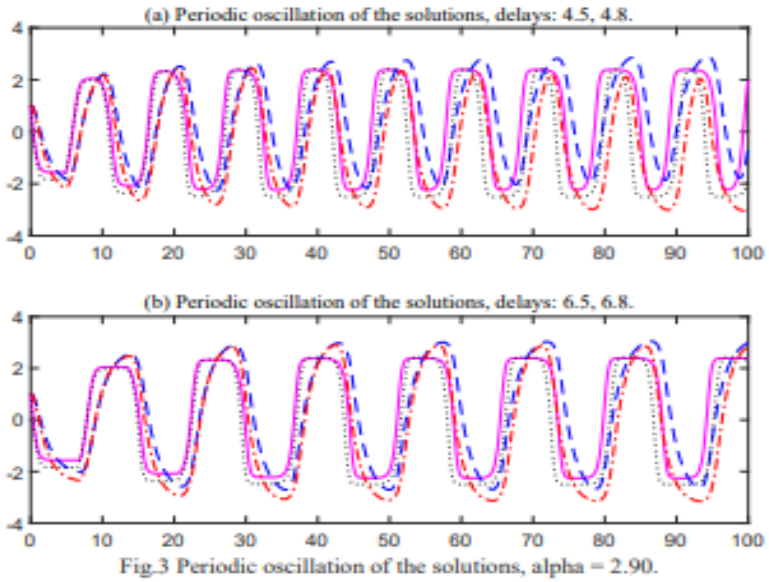


Fig.3 Periodic oscillation of the solutions, alpha = 2.90.

Then we only change α = 3.10, all other parameters are the same as in figure 3, we see that the periodic oscillation of the solutions appeared around the equilibrium point $(x_1^*, y_1^*, x_2^*, y_2^*)^T = (0.0152, 0.1637, -0.0306, -0.1808)^T$ (see Fig. 4).

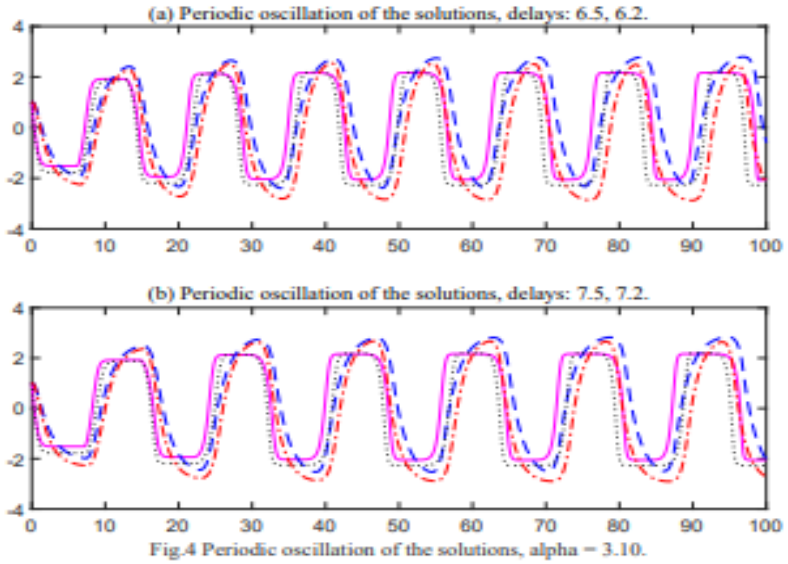


Fig.4 Periodic oscillation of the solutions, alpha = 3.10.

Finally, we select α = 3.00, η_{11} = 0.38, η_{12} = 0.55, η_{13} = 0.25, η_{21} = 0.35, η_{22} = 0.54, η_{23} = 0.28, a_1 = 0.58, a_2 = 0.56, k_1 = −1.25, k_2 = −1.15, b_1 = 0.75, b_2 = 0.85, c_1 = 0.72, c_2 = 0.75, Λ_1 = 0.15, Λ_2 = 0.18, we see that σ = 2.4454, and σ + k_2 = 1.2954 > 0, the condition of Theorem 2 is satisfied, there exists a periodic oscillatory solution around the equilibrium point $(x_1^*, y_1^*, x_2^*, y_2^*)^T = (0.2346, 0.3114, 0.2979, 0.2838)^T$ (see Fig. 5). From the figures, we see that time delays affect the oscillatory frequency.

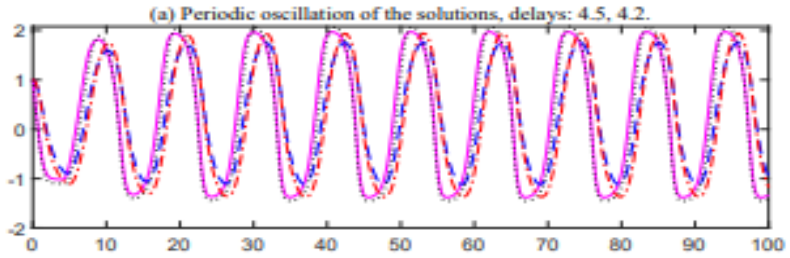


Fig.5 Periodic oscillation of the solutions, alpha = 3.00.

CONCLUSION

The present paper discusses an extended Boissonade-De Kepper model with discrete delays by means of the extended Chafee's criterion. Two sufficient conditions are provided to guarantee the existence of periodic solutions. We change the nonlinear model to an equivalent system. The instability of the solution of the equivalent system implies the instability of the equilibrium point of the original system. Our criterion is only a sufficient condition.

References

[1] J. Boissonade, P.D. Kepper, Transitions from bistability to limit cycle oscillations: theoretical analysis and experimental evidence in an open chemical system, J. Phys. Chem., 84 (1980) 501-506.

[2] D. Toker, F.T. Sommer, M. Esposito, A simple method for detecting chaos in nature, Commun. Biol., 3 (2020) 1-13.

[3] H. Niederholtmeyer, Z.Z. Sun, Y. Hori, E. Yeung, A. Verpoorte, R.M. Murray, S.J. Maerkl, Rapid cell-free forward engineering of novel genetic ring oscillators, Elife, 4 (2015)

[4] L.F. Olsen, H. Degn, Chaos in an enzyme reaction, Nature, 267 (1977) 177-178.

[5] A. Lekebusch, A. Forster, F.W. Schneider, Chaos control in an enzymatic reaction, J. Phys. Chem., 99 (1995) 681-686.

[6] I.R. Epstein, Y. Lao, Differential delay equations in chemical kinetics. Nolinear model: The cross-shaped phase diagram and the oregonator, J. Phys. Chem., 95 (1991) 244-254.

[7] A. Abdulrahuman, K.S. Chakrabarti, G. Raina, Stability, convergence and bifurcation of a reduced Boissonade-De Kepper model with delayed feedback, IFAC PaperOnLine 54-18 (2021) 76-83.

[8] J. Zhang, X. Gu, Stability and bifurcation analysis in the delay-coupled van der Pol oscillators, Appl. Math. Model., 34 (2010) 2291-2299.

[9] X.Y. Li, J.C. Ji, C.H. Hansen, Dynamics of two delay coupled van der Pol equations, Mech. Res. Commun., 33 (2006) 614-627.

[10] K. Miyamoto, K. Ryono, T. Oguchi, Delay-independent synchronization in networks of time delay coupled systems with uncertainties, IFAC PaperOnLine 51-33 (2018) 211-216.

[11] Y. Liu, J. Cai, H. Xu, M. Shan, Q. Gao, Stability and Hopf bifurcation of a love model with two delays, Math. Comput. Simul., 205 (2023) 558-580.

[12] P.V. Satyanarayana, S. Sivaprakasam, PT-symmetry in mutually coupled semiconductor lasers with a non-zero-time delay, Physics Lett. A, 452 (2022) 128449.

[13] E. Steur, W. Michiels, H. Huijberts, H. Nijmeijer, Networks of diffusively time-delay coupled systems: Conditions for synchronization and its relation to the network topology, Physica D, 277 (2014) 22-39.

[14] H. Achouri, C. Aouiti, B.B. Hamed, Codimension two bifurcation in a coupled FitzHugh Nagumo system with multiple delays, Chaos, Solitons and Fractals, 156 (2022) 111824.

[15] X. He, X. Li, S. Song, Finite-time stability of state-dependent delayed systems and application to coupled neural networks, Neural Networkls, 154 (2022) 303-309.

[16] J.K. Hale, S.M. Verduyn, Introduction to functional differential equations, New York, Springer, 1993.

[17] N. Chafee, A bifurcation problem for a functional differential equation of finite retarded type, J. Math. Anal. Appl., 35 (1971) 312-348.

[18] C. Feng, R. Plamondon, An oscillatory criterion for a time delayed neural network model, Neural Networks, 29 (2012) 70-79.