Transactions on Engineering and Computing Sciences - Vol. 11, No. 1
Publication Date: February 25, 2023
DOI:10.14738/tecs.111.13785.
Tong, M. (2023). Strong Maximal L^q- L^p-estimates for the Solution of the Stokes Equation. Transactions on Engineering and Computing Sciences, 11(1). 132-142.

Strong Maximal $L^q - L^p$ – estimates for the Solution of the Stokes Equation

Maoting Tong
Department of Mathematical Science, The University of Liverpool, U.K.,
Public Basic Department， Nanjing Vocational University of Industry Technology，
Nanjing, China

Abstract

Suppose that Ω is a bounded domain in R^3 with smooth boundary $\partial\Omega$ of class C^3. In this paper we prove that if the initial value $u_0 \in D((-\Delta)^{1/2})$, then there are strong maximal $L^q - L^p$ -estimates for the solution of the Stokes equation (1) for any $1<q<\infty$, $2 \le p<\infty$. Mathematics Subject Classification (2010). Primary 35Q30, 76D07; Secondary76N10, 47D06.

Keywords. Navier-Stokes equation, Existence and uniqueness, Global solution, Semigroup of operators, Invariance, Banach lattice, Fractional powers.

The Navier-Stokes initial value problem can be written in its classical form as

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u - \nabla p - (u \bullet \nabla)u + f, x \in \Omega, t \in (0,T) \\ \nabla \bullet u \equiv divu = 0 \\ u\big|_{\partial\Omega} = 0, t \in (0,T) \\ u\big|_{t=0} = u_0, x \in \Omega \end{cases}$$

where Ω is a bounded domain in R^3 with smooth boundary $\partial\Omega$ of class C^3 , $u = u(t, x) = (u_1(t,x),u_2(t,x),u_3(t,x))$ is the velocity field, $u_0 = u_0(x)$ is the initial velocity， $p = p(t, x)$ is the pressure, $f = f(t, x)$ is the external force,

$$\Delta u = (\sum_{i=1}^{3} \frac{\partial^2 u_1}{\partial x_i^2}, \sum_{i=1}^{3} \frac{\partial^2 u_2}{\partial x_i^2}, \sum_{i=1}^{3} \frac{\partial^2 u_3}{\partial x_i^2}).$$

In these four equations u, p are unknown and f , u_0 are given. The boundary condition imposed on the velocity at Ω is homogeneous. If the nonlinear term $(u\bullet\nabla)u=0$ we get the following Stokes equation with the classical Dirichlet boundary condition

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u - \nabla p + f, x \in \Omega, t \in (0,T) \\ \nabla \bullet u \equiv divu = 0 \\ u\big|_{\partial\Omega} = 0, t \in (0,T) \\ u\big|_{t=0} = u_0, x \in \Omega \end{cases} . \tag{1}$$

The existence， uniqueness and regularity properties of solutions for the Stokes equations are extensively studied. There is an extensive literature on the solvability of the initial value problem for Stokes equations.

Let L_p (Ω) ($2 \le p < \infty$) be the Banach space of real vector functions in L^P (Ω) . That is

$$L_p(\Omega) = \left\{h : \Omega \to R^3, h = (h_1, h_2, h_3), h_i \in L^p(\Omega)(i = 1,2,3)\right\}.$$

For $u=(u_1,u_2,u_3) \in L_P(\Omega)$, we define the norm

$$\|u\|_{L_p(\Omega)} = (\sum_{i=1}^{3} \|u_i\|_{L^p(\Omega)}^p)^{\frac{1}{p}}$$

then L_p (Ω) is a Banach space. The set of all real vector functions u such that $div\ u=0$and $u \in C_0^\infty(\Omega)$ is denoted by $C_{0,\sigma}^\infty(\Omega)$. Let DL (Ω) be the closure of $u \in C_0^\infty(\Omega)$ in L(Ω).If $u \in C^\infty(\Omega)$ then $u \in DL(\Omega)$ implies $divu=0$.(seep.270in [3]). Similarly if $u \in C^\infty$ (Ω) then

$$u \in DL_p(\Omega) \subset DL_2(\Omega) \text{impli } div\ u=0. \tag{2}$$

In this paper we always consider the spaces of vector value functions on Ω . Some authors den ote DL_p (Ω) by $L^p{}_\sigma$ (Ω). From [3],[7] and [10,p.129] we have the Helmholtz decomposition

$$L_p(\Omega) = DL_p(\Omega) \oplus DL_p(\Omega)^\perp$$

where

$$DL_p(\Omega)^\perp = G_p(\Omega) = \left\{u \in L_p(\Omega) : u = \nabla \pi \quad \text{for some} \quad \pi \in W_{loc}^{1,p}(\Omega)\right\}.$$

Let P be the orthogonal projection from L_p (Ω) onto DL_p (Ω). By applying

P to the first equation of (1) and taking account of the other equations , we are let the following abstract initial value problem, Pr . II

$$\begin{cases} \dfrac{du}{dt} = P\Delta u + Pf, t \in (0,T) \\ u|_{t=0} = u_0, x \in \Omega \end{cases}. \tag{3}$$

We consider equation(3)in integral form Pr.III

$$u(t) = e^{tP\Delta}u_0 + \int_0^t e^{(t-s)P\Delta}Fu(s)ds. \tag{4}$$

The projection operator P is called the Helmholtz projection. An important tool in order to handle the incompressible condition *div* $u = 0$ is the Helmholtz decomposition. In the simplest case, where $\Omega \subset R^3$ is a domain with compact and smooth boundary, the operator A_p given by

$$A_p u := P\Delta u, D(A_p) := W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \cap DL_p(\Omega)$$

is called the Stokes operator.([10.p.119]).

By maximal $L^q - L^p$ – estimates for the solution of the Stokes equation , one understands the following: given1<p,q<∞ and $f \in L^q((0,\ T);DL_p(\Omega))$ does there exist a unique $u \in L^q((0,T);D(A_p))$ with $\partial_t u \in L^q((0,T);DL_p(\Omega))$ satisfying

$$\begin{cases} \dfrac{\partial u}{\partial t} = P\Delta u + f, t \in (0,T) \\ u(0) = u_0 \end{cases},$$

and a constant C >0, independent of f, such that

$$\|\partial_t u\|_{L^q((0,T);L_p(\Omega))} + \|A_p u\|_{L^q((0,T);L_p(\Omega))} \le C\|f\|_{L^q((0,T);L_p(\Omega))}.$$

It was shown by Geissert, Heck, Hieber,and Sawada [4] that the existence of the Helmholtz decomposition implies the existence of the Stokes semigroup on $L^P{}_\sigma(\Omega)$ and that the Stokes operator admits maximal regularity for domains Ω with uniform C^3 – boundaries $\partial\ \Omega$ (but possible non-compact) and $u_0 = 0$. It is known that the Helmholtz decomposition exists for p = 2 for arbitrary open sets Ω. But note that due to the results by Maslennikova and Bogovoskii [14], the Helmholtz decomposition for L_p (Ω) does not exist in general if $p \neq 2$. Hence, a characterization of those domains, even with smooth boundaries , for which the solution of the

Stokes equation is governed by an analytic semigroup on DL_p (Ω) remains a challenging open problem until now.

The existence of the Helmholtz decomposition is not necessary for the existence of the Stokes semigroup by Bolkart, Giga, Miura, Suzuki, and Tsutsui [2] shows. Maximal $L^p - L^p$ – estimates for bounded or exterior domains with smooth boundaries were proved first by Solonnikov [21], while maximal mixed $L^p - L^q$ – estimates were obtained first by Giga and Sohr [9] for bounded or exterior Ω and $u_0 \in D_q^{1-1/s-s}$. In [5] M.Geissert, M.Hess, M.Hieber, C.Schwarz, K.Stavrakidis gave a short and new proof for maximal $L^p - L^q$ -estimates for bounded or exterior Ω with compact C^3 -boundary $\partial\Omega$ and $u_0 = 0$. Nowweintroduceanewconcept.Let $C^{\mu}((0,T);X)$ denote the space of Hölder continuous functions on $(0,T)$ with exponent μ and with values in a Banach space X. By strong maximal $L^q - L^p$ – estimates for the solution of the Stokes equation ,one under stands the following:given1<p,q<∞ and $f \in C^{\mu}((0,T);DL_p(\Omega))$ does there exist a unique strong solution $u \in L^q((0,T);D(A_p))$of the Stokes equation

$$\begin{cases} \dfrac{\partial u}{\partial t} = P\Delta u + f, t \in (0,T) \\ u(0) = u_0 \end{cases},$$

with $\partial_t u \in L^q((0,T);DL_p(\Omega))$, and a constant C>0,independent of f, such that

$$\|\partial_t u\|_{L^q((0,T);L_p(\Omega))} + \|A_p u\|_{L^q((0,T);L_p(\Omega))} \le C\|f\|_{L^q((0,T);L_p(\Omega))}.$$

In this paper we will prove that the solution of the Stokes equation (1) is governed by an analytic semigroup on $L^p{}_\sigma$ (Ω) for bounded domain Ω in R^3 with smooth boundary $\partial\Omega$ of class C^3 by using the theory of semigroup of operators. If the initial value $u_0 \in D((-\Delta)^{\frac{1}{2}})$ then there exists a strong maximal $L - L$ – estimates for the solution of the Stokes equation for bounded domains Ω with smooth boundary. Although $f \in C^{\mu}((0,T);DL_p(\Omega))$ implies $f \in L^q((0\ T);DL_p(\Omega))$, our result is always different from the previous result: u is the strong solution of (3), $\partial\Omega$ is smooth but does not need to be compact and $u_0$does not need to be zero.

For $u = (u_1, u_2, u_3) \in L_p(\Omega)$ we define $\Delta u = (\Delta u_1, \Delta u_2, \Delta u_3)$ and $\nabla u = \left(\frac{\partial u_1}{\partial x_1}, \frac{\partial u_2}{\partial x_2}, \frac{\partial u_3}{\partial x_3}\right)$. Since the operator $-\nabla = -\sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2}$ is strongly elliptic of order 2.

Theorem7.3.6 in [17] implies that Δ is the infinitesimal generator of an analytic semigroup of contractions on $L^p(\Omega)$ with $D(\Delta) = W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega)$. Hence Δ is also the infinitesimal generator of an analytic semigroup of contraction on $L_p(\Omega)$ with $D(\Delta) = W^{2p}(\Omega) \cap W^{1}{}_{0}{}^{p}(\Omega)$, where $W^{2p}(\Omega)$ and $W^{1}{}_{0}{}^{p}(\Omega)$ are the Sobolev spaces of vector value in $W^{2,p}(\Omega)$and $W_0^{1,p}(\Omega)$ respectively.We will prove that Δ is also the infinitesimal generator of an analytic semigroup of contraction on $DL_p(\Omega)$.

A operator A is called preserving divergence-free on a vector value functions space X if A map severy $u \in X$ with $div\ u=0$ toan Au with $div\ Au=0$.

Lemma 1.(Lemma 1 in [22]) For every $u \in L_p(\Omega)$, $div\ u = 0$ if and only if $div(\lambda I - \Delta)u=0$.

for

$$\lambda \in \Sigma_\vartheta = \{\lambda : \vartheta - \pi < \arg\lambda < \pi - \vartheta, \lambda \geq r\}$$

where $0<\vartheta<\frac{\pi}{2}$

Lemma2.(1.5.12in[5])Let $\{T(t): t\geq 0\}$ bea C_0 -semi group on a Banach space X.If Y is a closed subspace of X such that $T(t)Y \subset Y$ for all $t\geq 0$,i.e.,if

$Y$is $T(t)_{t\geq 0}$-invariant,then the restrictions

$$T(t)_{|} := T(t)_{|Y}$$

form a C_0 -semigroup $\{T(t) : t \geq 0\}$, called the subspace semigroup, on the Banach space Y.

Lemma 3. (Proposition 2.2.3 in [5]) Let $(A, D(A))$ be the generator of a C_0 -semigroup $\{T(t) : t \geq 0\}$on a Banach space X and assume that the restricted semigroup (subspace semigroup) $\{T(t) : t \geq 0\}$ is a C_0 -semigroup on some $(T(t))_{t\geq 0}$ – invariant Banach space $Y \to X$. Then the generator of $\{T(t) : t \geq 0\}$ is thepart $(A, D(A))$ of A in Y.

Lemma 4. The operator Δ with $D(\Delta) \subset DL_p(\Omega)$ is the infinitesimal generator of an analytic semigroup of contractions on $DL_p(\Omega)$.

Proof. From Theorem 7.3.6 in [17] Δ is the infinitesimal generator of an analytic semigroup of contractions on $L^p(\Omega)$. Then Δ is also the infinitesimal generator of an analytic semigoup of contractions on $L_p(\Omega)$. Let $\{T(t)\ t \geq 0\}$ be the restriction of the analytic semigroup generated by Δ on $L_p(\Omega)$ to the real axis . $\{T(t)\ t \geq 0\}$is a C_0 semigroup of contractions by Theorem 7.2.5 and Theorem 3.1.1 in [17]. We have already noted that $DL_p(\Omega)$ is a closed subspace of $L_p(\Omega)$ and is also a Banach space. We want to show that $DL_p(\Omega)$ is $T(t)_{t\geq 0}$ – invariant. For every $u \in L_p(\Omega)$with $div\ u=0$ and$\lambda \in \rho(\Delta) \cap \Sigma_\vartheta = \{\lambda : \vartheta - \pi < \arg\lambda < \pi - \vartheta, \lambda \geq r\}$we have $(\lambda I - \Delta)[R(\lambda : \Delta)u] = u$

where $\sum_{\vartheta}$ is the same as in the proof of lemma 1. From Lemma 1 it follows that *div* $R(\lambda:\Delta)u=0$. That is to say that $R(\lambda:\Delta)$ is preserving divergence-free for $\lambda \in \rho(\Delta) \cap \Sigma_{\vartheta}$. From Theorem 2.5.2 (c) in [17] it follows that $\rho(\Delta) \supset R^{+}$, and so $\rho(\Delta) \cap \Sigma_{\vartheta} \supset \{\lambda:\lambda \geq r\}$. Hence $R(\lambda:\Delta)$ is preserving divergence free for every $\lambda \geq r$. Let $u \in DL_p(\Omega)$ then there exists a sequence u_n such that $\lim_{n\to\infty} u_n = u$ and div $u_n = 0$ for $n = 1,2,\ldots$. Since $R(\lambda : \Delta)$ is bounded and so is continuous. Hence $\lim_{n\to\infty} R(\lambda:\Delta)u_n = R(\lambda:\Delta)u$ and $div\, R(\lambda:\Delta)u_n = 0$ for every $\lambda \geq r$. Therefore $R(\lambda:\Delta)u \in DL_p(\Omega)$ for every $\lambda \geq r$. It follows that $DL_p(\Omega)$ is $R(\lambda : \Delta)$ -invariant for every $\lambda \geq r$. Now the Theorem 4.5.1 in [17] implies that $DL_p(\Omega)$ is $T(t)_{t\geq 0}$ − invariant. From Lemma 2 and Lemma 3 it follows that $\Delta\, DL_p(\Omega)$with $D(\Delta DL_p(\Omega))=D(\Delta)\cap DL_p(\Omega)$ is the infinit esimal generator of the C_0 semigroup $\{T(t)\ DLp(\Omega):t\geq 0\}$of contractions on $DL_p(\Omega)$.

We will prove that $\{T(t)\ DL_p(\Omega):t\geq 0\}$ can also be extended to an analytic semigroup on

on $DL_p(\Omega)$. Suppose that $\lambda \in \rho(\Delta) \cap \Sigma_{\vartheta}$, i.e. there exists $R(\lambda:\Delta)$ from $L_p(\Omega)$ into$D(\Delta)$. Then for any $u \in DL_p(\Omega) \subset L_p(\Omega)$, $R(\lambda:\Delta)u \in DL_p(\Omega)$. We have

$$(\lambda I-\Delta)R(\lambda:\Delta)u=u,\ R(\lambda:\Delta)(\lambda I-\Delta)u=u. \qquad (5)$$

Since $R(\lambda : \Delta)$ is preserving divergence-free we have

$$R(\lambda:\Delta)u \in DL_p(\Omega) \text{ and } (\lambda I-\Delta)u \in DL_p(\Omega).$$

Thus the formula (5) becomes

$$(\lambda I - \Delta_{|DL_p(\Omega)})R(\lambda:\Delta)u = u \quad \text{and} \quad R(\lambda:\Delta)(\lambda I - \Delta_{|DL_p(\Omega)})u = u.$$

Hence $\left(\lambda I - \Delta_{|DL_p(\Omega)}\right)R(\lambda:\Delta) = I$ and $R(\lambda:\Delta)\left(\lambda I - \Delta_{|DL_p(\Omega)}\right) = I$. We get

$$(\lambda I - \Delta_{|DL_p(\Omega)})^{-1} = R(\lambda:\Delta)_{|DL_p(\Omega)}. \qquad (6)$$

From the formula (6) and Theorem 2.5.2(c) in [17] we have

$$\rho(\Delta\, DL_p(\Omega)) \supset \rho(\Delta) \cap \Sigma_{\vartheta} \supset \Sigma = (\{\lambda:\arg\lambda < \pi_2 + \delta\} \cup \{0\}) \cap \Sigma_{\vartheta}$$
$$= \Sigma_{\vartheta_1} = \{\lambda:\arg\lambda < \vartheta_1, \lambda \geq r\}$$

where $0 < \delta < \pi/2$ and $\vartheta_1 = min\{\pi/2 + \delta, \pi - \vartheta\}$. Thus, for $\lambda \in \Sigma_{\vartheta_1}$, $\lambda I - \Delta_{|DL_p(\Omega)}$ is

invertible. From Theorem 2.5.2(c) in [17] we have

$$\left\| R(\lambda : \Delta_{|DL_p(\Omega)} \right\| = \underset{\substack{u \in DL_p(\Omega) \\ \|u\|_{DL_p(\Omega)}=1}}{Sup} \left\| R(\lambda : \Delta_{|DL_p(\Omega)}) u \right\|$$

$$\leq \underset{\substack{u \in L_p(\Omega) \\ \|u\|_{Lp(\Omega)}=1}}{Sup} \left\| R(\lambda : \Delta) u \right\| = \left\| R(\lambda, \Delta) \right\|_{L_p(\Omega)} \leq \frac{M}{|\lambda|}.$$

Now Theorem 2.5.2(c) in [17] implies that $\left\{ T(t)_{|DL_p(\Omega)} : \lambda \geq 0 \right\}$ can also be extended to an analytic semigroup on $DL_p(\Omega)$. Therefor $\Delta_{DL_p(\Omega)}$ is a infinitesimal generator of an analytic semigroup of contraction on $DL_p(\Omega)$. □

Lemma5. $A_p = \Delta_{DL2(\Omega)}$ on $DL_p(\Omega)$.
Proof. We have

$$D(\Delta DL_p(\Omega)) = D(\Delta) \cap DL_p(\Omega) \text{ and } D(A_p) := W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \cap DL_p(\Omega).$$

Hence

$$D(\Delta DL_p(\Omega)) = D(A_p).$$

On the other hand, we proved in lemma 4 that $\Delta_{DL_p(\Omega)}$ with is the infinitesimal generator of the C_0 semigroup $\{T(t)_{/DLp(\Omega)} : t \geq 0\}$ of contractions on $DL_p(\Omega)$. That is, $DL_p(\Omega)$is a invariant subspace under $\Delta_{/DLp(\Omega)}$. And $A_p u := P\Delta u$. So for any $u \in D(\Delta_{DLp(\Omega)}) = D(A_p)$ $\Delta u \in DL_p(\Omega)$. But P is the orthogonal projection from $L_p(\Omega)$ onto $DL_p(\Omega)$. So

$$P\Delta u = \Delta u.$$

Finally we have

$$A_p = \Delta_{DL_p(\Omega)}. \square$$

Theorem 2 in [6] is similar the above Lemma 5. Sobolevskii proved the fact that $A_p = P\Delta$ generates an analytic semigroup on $L_\sigma{}^p(\Omega) = DL_p(\Omega)$in [20] . Giga gave a different proof in [6]. Our proof using the theory of semigroups of bounded operators is more simple.

Suppose that $-A$ is the infinitesimal generator of an analytic semigroup $T(t)$ on a Banach space X. From the results of section 2.6 in [17] we can define the fraction powers A^α for $0 \leq \alpha \leq 1$ and A^α is a closed linear invertible operator with domain $D(A^\alpha)$ dense in X. $D(A^\alpha)$ equipped with the

norm $\|x\|_\alpha = \|A^\alpha x\|$ is a Banach space denoted by X_α. It is clear that $0<\alpha<\beta$ implies $X_\alpha \supset X_\beta$ and that the embedding of X_β into X_α is continuous.

Assumption (F). Let $X = DL_p(\Omega)$ and U be an open subset in $R^+ \times X_\alpha$ $(0<\alpha<1)$. The function $f: U \to X$ satisfies the assumption (F) if for every $(t,u) \in U$ there is a neighborhood $V \subset U$ and constants $L \geq 0$, $0<\vartheta \leq 1$ such that for all $(t_i,u_i) \in V (i=1,2)$

$$\|f(t_1,u_1) - f(t_2,u_2)\| \leq L(|t_1 - t_2|^\vartheta + \|u_1 - u_2\|_\alpha)$$

Lemma 6. (Theorem 6.3.1 in [17]) Let $-A$ be the generator of an analytic semigroup $T(t)$ satisfying $\|T(t)\| \leq M$ and assume that $0 \in \rho(-A)$. If, $0<\alpha<1$ and f satisfies the assumption (F) then for every initial date $(t_0, u_0) \in U$ the initial value problem

$$\begin{cases} \dfrac{du(t)}{dt} + Au(t) = f(t,u(t)), t \in (t_0, \infty) \\ u(t_0) = u_0 \end{cases} \tag{7}$$

has a unique local solution $u \in C([t_0,t_1): X) \cap C^1((t_0,t_1): X)$ where $t_1 = t_1(t_0,u_0) > t_0$.
Now we study the Stokes equation (1).

A function u which is differentiable almost everywhere on $[0, T]$ such that $u \in L^1[0,T: DL_2(\Omega)]$ is called a strong solution of the initial value problem (1) if $u(0) = u_0$ and u satisfies (1) a.e. on $[0,T]$.

Theorem 1. Suppose that Ω is a bounded domain in R^3 with smooth boundary $\partial\Omega$ of class C^3. The Stokes equation (1) has a unique local strong solution if the initial value $u_0 \in D((-\Delta)^{1/2})$, $f \in C^\beta((0,T); DL_p(\Omega))$, i.e. there exist constant C and $0 < \beta < 1$ such that

$$\|f(s,x) - f(t,x)\|_{DL_p(\Omega)} \leq C|s-t|^\beta, \tag{8}$$

for $s,t \in (0,T)$ and $x \in \Omega$.

Proof. We will find that by incorporating the divergence-free condition, we can remove the pressure term from our equation. (see p. 271 in [3], p. 234_6 and p. 239_9 in (9). So first we can rewrite (1) into a abstract initial value problem on DL_p (Ω)

$$\begin{cases} \dfrac{du}{dt} = \Delta u + F(t,u(t)), t \in (0,T) \\ u\big|_{t=0} = u_0, x \in \Omega \end{cases} \tag{9}$$

where $F(t,u(t)) = f$ is a abstract function. From Lemma 4 Δ is the generator of an analytic semigroup T (t) of contraction on DL_p (Ω) and $//T(t) \le 1//$. From Theorem 2.5.2(c) in [17] 0 ∈ ρ (Δ).

Let $U=R^+ \times DL_p(\Omega)_{1/2}$.Then U is a open set and $F(t,u(t))=f(t)$is a function: $U \to DL_p(\Omega)$. Since $u_0 \in D((-\Delta)^{1/2})=DL_p(\Omega)_{1/2}$, $(0,u_0) \in U$.

From(8)itfollowsthatforall $(t_i,u_i) \in U(i=1,2)$

$$\begin{aligned} &\left\| F(t_1,u_1(t_1)) - F(t_2,u_2(t_2)) \right\|_{DL_p(\Omega)} \\ &\le \left\| f(t_1,x) - f(t_2,x) \right\|_{DL_p(\Omega)} \\ &\le C\left| t_1 - t_2 \right|^{\beta} \end{aligned}$$

Hence $F(t,u(t))$ satisfies the assumption (F) , then by Lemma 6 for every initial data $(0,u_0) \in U$ the initial value problem (9) has a unique local solution

$$u(t) \in C\left([0,t_1) : DL_p(\Omega)\right) \cap C^1\left((0,t_1) : DL_p(\Omega)\right) \tag{10}$$

where$t_1 = t_1(u_0)$.

In a similar induction way as Theorem 3.9 in [2] or as Theorem 5.1 in [19] we can prove that the solution $u(t,x) \in \left(C^{\infty}\left((0,t_1)\times\overline{\Omega}\right)\right)^3$. We can also prove directly that $u(t,x)$ is smooth. In fact, the solution (10) of (9) is also the solution of (4). The Theorem 3.4 in [2] mean that as long as the solution of (4) exists , this solution is smooth. From Theorem 3.4 in [2] we have the solution $u(t,x) \in \left(C^{\infty}\left((0,t_1)\times\overline{\Omega}\right)\right)_3$. Substituting $u(t,x)$into(1)we get the solution $p(t,x)$.We also have $p(t,x) \in$

$C^{\infty}((0,t_1)\times\Omega)$. It follows from the formula (2) and $u,p\in DL_p(\Omega)$ that the solution $u(t,x)$ and $p(t,x)$ are divergence-free. Changing the value of u on $\partial\Omega$ to zero we get a unique local strong solution $u=u(t,x)$, $p(t,x)$ of the Stokes equation(1) □

From(1) we see that $u(t)\in D(\Delta)=W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)\cap DL_p(\Omega)$. From(10) $u(t)\in C([0,t_1):DL_p(\Omega))$ and $u'(t)\in C((0,t_1):DL_p(\Omega))$forany $1<q<\infty$.Hence $u(t)\in L^q((0,t_1):D(A_p))$ and $u'\in L^q((0,t_1):DL_p(\Omega))$ for any $1<q<\infty$. From(9)we have

$$u'-\Delta u=f.$$

Hence we get

$$\left\|u'\right\|_{L^q((0,t_1):L^p(\Omega))}+\left\|A_pu\right\|_{L^q((0,t_1):L^p(\Omega))}\le\left\|f\right\|_{L^q((0,t_1):L^p(\Omega))}.$$

Therefore we have

Theorem 2. Suppose that Ω is a bounded domain in R^3 with smooth boundary $\partial\Omega$ of class C^3. If the initial value $u_0\in D((-\Delta)^{1/2})$, then there are strong maximal L^q-L^p -estimates for the solution of the Stokes equation (1) for any $1<q<\infty$, $2\le p<\infty$.

References

[1] Andras Batkai, Marjeta Kramar Fijavz, Abdelaziz Rhandi. Positive Operator Semigroups, Birkhauser (2017)

[2] M. Bolkart, Y. Giga, T. Miura, T. Suzuki, Y. Tsutsui, On analyticity of the L^p -Stokes semigroup for some non-Helmholtz domains. Math. Nachrichten. (to appear)

[3] H.Fujita, T.Kato, On the Navier-Stokes initial value problem I, Arch. Ration. Mech. Anal. 16(1964),269-315

[4] M.Geissert, H.Heck, M.Hieber, O.Sawada, Weak Neumann implies Stokes. J.Reine Angew. Math. 669, 75–100 (2012)

[5] M.Geissert, M.Hess, M.Hieber, C.Schwarz, K.Stavrakidis, Maximal L^p-L^q -estimates for the Stokes equation: a short proof of Solonnikov's Theorem, J. Math. Fluid Mech. 12, 47–60 (2010)

[6] Y. Giga, Analyticity of the semigroup generated by the Stokes operator on L_r -spaces, Math. Z. 178, 297–329 (1981)

[7] Y.Giga,T.Miyakava, Solutions in L_r of the Vavier-Stokes Initial Value Problem, Arch.Ration. Mech.Anal. 89(1985) 267-281

[8] Y. Giga, Domains of fractional powers of the Stokes operator in L_r spaces, Arch. Ration. Mech. Anal. 89, 251–265 (1985)

[9] Y. Giga, H. Sohr, Abstract L^p – estimates for the Cauchy problem with applications to the Navier-Stokes equations in exterior domains, J. Funct. Anal. 102, 72–94 (1991)

[10] Y.Giga, A.Novotny, Hamdbook of Mathematical Analysis in Mechanics of Viscous Fluids, Springer, 2018

[11] T.Kato, Strong L_p -solution of the Navier-Stokes equation in R^m , with application to weak solutions, Math. Z. 187(1984)471-480

[12] Engel Klaus-Jochen , Rainer Nagel, One-Parameter Semigroups for Linear Evolution Equations, Springer, (2000)

[13] E.Kreyszig, Introduction functional analysis with applications, John Wiley &Sons (1978)

[14] V. Maslennikova, M. Bogovskii, Elliptic boundary values in unbounded domains with noncompact and nonsmooth boundaries, Rend. Sem. Mat. Fis. Milano 56, 125– 138 (1986)

[15] Peter Meyer-Nieberg, Banach Lattices, Springer-Verlag, (1991)

[16] M.Otelbaev, Existence of a strong solution of the Navier-Stokes equation, Mathematical Journal, 13(4)(2013),5-104

[17] A.Pazy, Semigroups of linear operators and applications to partial differential equations, Springer Verlag (1983,reprint in China in 2006)

[18] James C.Robinson, Infinite-Dimentional Dynamical Systems, Cambridge University Press, (2001)

[19] Veli B.Shakhmurov, Nonlocal Navier-Stokes problems in abstract function, Nonlinear Analysis:Real World Applications , 26(2015)19-43

[20] P.E. Sobolevskii, Study of the Navier-Stokes equations by the methods of the theory of parabolic equations in Banach spaces. Sov. Math. Dokl. 5, 720–723 (1964)

[21] V.A. Solonnikov, Estimates for solutions of nonstationary Navier-Stokes equations, J. Sov. Math. 8, 467–529 (1977)

[22] Maoting Tong, Daorong Ton, A Local Strong Solution of the Navier-Stokes Problem in L_2 (Ω), Journal of Mathematical Sciences: Advances and Applications,Volume 62, 2020, Pages 1-19,Available at http://scientificadvances.co.in DOI: http://dx.doi.org/10.18642/jmsaa_7100122125

[23] K.Yosida, Functional analysis (6^{th} edition), Springer Verlag (1980)