Transactions on Machine Learning and Artificial Intelligence - Vol. 9, No. 4
Publication Date: August, 25, 2021
DOI:10.14738/tmlai.94.10523.
Alpatov, A., Kravets, V., Kravets, V., & Lapkhanov, E. (2021). Representation of the Kinematics of the Natural Trihedral of a Spiral-Helix Trajectory by Quaternion Matrices. Transactions on Machine Learning and Artificial Intelligence, 9(4). 18-29.

Representation of the Kinematics of the Natural Trihedral of a Spiral-Helix Trajectory by Quaternion Matrices

Anatolii Alpatov
The Institute of Technical Mechanics of the
National Academy of Sciences of Ukraine and the
State Space Agency of Ukraine, Dnipro, Ukraine

Victor Kravets
Department of Mechanics
Dnipro University of Technology, Dnipro, Ukraine

Volodymyr Kravets
Department of Mechanics
Dnipro State Agrarian and Economic University, Dnipro, Ukraine

Erik Lapkhanov
The Institute of Technical Mechanics of the
National Academy of Sciences of Ukraine and the
State Space Agency of Ukraine, Dnipro, Ukraine

ABSTRACT

The spiral-helix trajectory of the transport vehicle programmed motion in the form of a hodograph in the stationary frame of reference is considered. A relative frame of reference associated with the natural trihedral of the trajectory is introduced. The formulas of curvature and torsion of the trajectory, the unit vector of the natural trihedral, the components of the angular velocity of rotation of the natural trihedral in the proper axes and in the stationary frame of reference are set in the quaternionic matrices. The results are verified using the Frenet-Serret formulas. The mathematical apparatus of quaternion matrices is tested with the aim of adapting spatial, nonlinear problems of dynamic design of transport vehicles to a computational experiment.

Keywords: Quaternion matrices; hodograph; spiral-helix trajectory, natural trihedral, kinematics, curvature, torsion, Frenet- Serret formulas.

INTRODUCTION

In the process of dynamic design, experimental and natural development, transport crews, including unmanned vehicles, are required to ensure controllability and maneuverability, stability and stabilization, orientation and navigation, reliability and safety [1-6]. Dynamic problems of the class under consideration are investigated in a computational experiment using mathematical models of nonlinear kinetics, kinematics, orientation, navigation [7-16]. The synthesis of an adequate mathematical model, adapted to computer technologies and describing spatial transfer and rotation, is an urgent problem of applied mechanics and

mathematics [17-19]. The developed mathematical apparatus is applied to the orientation of the relative frame of reference [20]. Here the efficiency of applying quaternion matrices to the description of the spatial rotation of the natural trihedral of a spiral-helix trajectory is illustrated. The components of the angular velocity of rotation of the natural trihedron are found in the proper, relative axes and in a stationary, inertial frame of reference.

PROBLEM STATEMENT

The programmed motion of the transport crew is set in a stationary frame of reference by the hodograph of the spiral-helix trajectory [21] in the follow form:

$$\bar{r}(t) = \left\| \rho_0 \quad \rho_1 \quad \rho_2 \quad \rho_3 \right\| \begin{Vmatrix} 1 \\ t \\ t^2 \\ t^3 \end{Vmatrix} \left(\bar{i}\cos(\omega t) + \bar{j}\sin(\omega t) \right) + \bar{k} \left\| h_0 \quad h_0 \quad h_0 \quad h_0 \right\| \begin{Vmatrix} 1 \\ t \\ t^2 \\ t^3 \end{Vmatrix} \quad (1)$$

where $\rho_i, h_i \ (i = 0,1,2,3)$ – variable parameters determined by specified boundary conditions;

t – time (independent parameter);

ω – average angular velocity of rotation, as

$$\omega = \frac{\varphi_0}{t_0}$$

φ_0 – full swing angle;

t_0 – required travel time in space;

$\bar{i}, \bar{j}, \bar{k}$ – basis of terrestrial cartesian coordinate system.

The unit vectors of the natural trihedral of the trajectory are determined by the following well-known formulas depending on the arc length *s* [22]:

$$\bar{\tau} = \frac{d\bar{r}}{ds}, \quad \bar{n} = \frac{1}{K}\frac{d^2\bar{r}}{ds^2}, \quad \bar{b} = \frac{1}{K}\frac{d\bar{r}}{ds} \times \frac{d^2\bar{r}}{ds^2}, \quad (2)$$

where K – curvature of trajectory:

$$K = \left| \frac{d\bar{\tau}}{ds} \right|,$$

or depending on scalar argument of time

$$\bar{\tau} = \frac{\dot{\bar{r}}}{\left|\dot{\bar{r}}\right|}, \quad \bar{n} = -\frac{\dot{\bar{r}} \times \left(\dot{\bar{r}} \times \ddot{\bar{r}}\right)}{\left|\dot{\bar{r}}\right| \left|\dot{\bar{r}} \times \ddot{\bar{r}}\right|}, \quad \bar{b} = \frac{\dot{\bar{r}} \times \ddot{\bar{r}}}{\left|\dot{\bar{r}} \times \ddot{\bar{r}}\right|}, \quad (3)$$

where $\left|\dot{\bar{r}}\right|^2 = \dot{\bar{r}} \cdot \dot{\bar{r}}$, $\left|\dot{\bar{r}} \times \ddot{\bar{r}}\right|^2 = \left(\dot{\bar{r}} \times \ddot{\bar{r}}\right) \cdot \left(\dot{\bar{r}} \times \ddot{\bar{r}}\right)$.

Here $\dot{\bar{r}}$ and $\ddot{\bar{r}}$ are the first and second time derivatives of the programmed hodograph. In matrix form [20], it can be written as:

$$\overline{\tau}=\frac{1}{|\dot{\bar{r}}|}\begin{Vmatrix}0\\ \dot{r}_1\\ \dot{r}_2\\ \dot{r}_3\end{Vmatrix},\quad \overline{n}=-\frac{1}{|\dot{\bar{r}}||\dot{\bar{r}}\times\ddot{\bar{r}}|}\begin{Vmatrix}0\\ \ddot{r}_1(\dot{r}_3\dot{r}_3+\dot{r}_2\dot{r}_2)-\dot{r}_1(\ddot{r}_2\dot{r}_2+\ddot{r}_3\dot{r}_3)\\ \ddot{r}_2(\dot{r}_3\dot{r}_3+\dot{r}_1\dot{r}_1)-\dot{r}_2(\ddot{r}_1\dot{r}_1+\ddot{r}_3\dot{r}_3)\\ \ddot{r}_3(\dot{r}_2\dot{r}_2+\dot{r}_1\dot{r}_1)-\dot{r}_3(\ddot{r}_1\dot{r}_1+\ddot{r}_2\dot{r}_2)\end{Vmatrix},\quad \overline{b}=\frac{1}{|\dot{\bar{r}}\times\ddot{\bar{r}}|}\begin{Vmatrix}0\\ \dot{r}_2\ddot{r}_3-\dot{r}_3\ddot{r}_2\\ \dot{r}_3\ddot{r}_1-\dot{r}_1\ddot{r}_3\\ \dot{r}_1\ddot{r}_2-\dot{r}_2\ddot{r}_1\end{Vmatrix}, \tag{4}$$

where:

$$\dot{r}_1=\begin{Vmatrix}\rho_0 & \rho_1 & \rho_2 & \rho_3\end{Vmatrix}\left[\begin{Vmatrix}0\\1\\2t\\3t^2\end{Vmatrix}\cos\omega t-\omega\begin{Vmatrix}1\\t\\t^2\\t^3\end{Vmatrix}\sin\omega t\right];$$

$$\dot{r}_2=\begin{Vmatrix}\rho_0 & \rho_1 & \rho_2 & \rho_3\end{Vmatrix}\left[\begin{Vmatrix}0\\1\\2t\\3t^2\end{Vmatrix}\sin\omega t+\omega\begin{Vmatrix}1\\t\\t^2\\t^3\end{Vmatrix}\cos\omega t\right];$$

$$\dot{r}_3=\begin{Vmatrix}h_0 & h_1 & h_2 & h_3\end{Vmatrix}\begin{Vmatrix}0\\1\\2t\\3t^2\end{Vmatrix};$$

$$\ddot{r}_1=\begin{Vmatrix}\rho_0 & \rho_1 & \rho_2 & \rho_3\end{Vmatrix}\left[\begin{Vmatrix}-\omega^2\\-\omega^2t\\2-\omega^2t^2\\6t-\omega^2t^2\end{Vmatrix}\cos\omega t-2\omega\begin{Vmatrix}0\\1\\2t\\3t^2\end{Vmatrix}\sin\omega t\right];$$

$$\ddot{r}_2=\begin{Vmatrix}\rho_0 & \rho_1 & \rho_2 & \rho_3\end{Vmatrix}\left[\begin{Vmatrix}-\omega^2\\-\omega^2t\\2-\omega^2t^2\\6t-\omega^2t^2\end{Vmatrix}\sin\omega t-2\omega\begin{Vmatrix}0\\1\\2t\\3t^2\end{Vmatrix}\cos\omega t\right];$$

$$\ddot{r}_3=\begin{Vmatrix}h_0 & h_1 & h_2 & h_3\end{Vmatrix}\begin{Vmatrix}0\\0\\2\\6t\end{Vmatrix}.$$

,

For a given hodograph of the programmed motion, it is required to find the components of the angular velocity vector of rotation of natural trihedral of the spiral-helix trajectory in its own relative axes $M\overline{\tau}\overline{n}\overline{b}$:

$$\overline{\Omega}=\overline{\tau}\Omega_\tau+\overline{n}\Omega_n+\overline{b}\Omega_b, \tag{5}$$

And in stationary frame of reference $O\bar{i}\bar{j}\bar{k}$:

$$\bar{\Omega} = \bar{i}\,\Omega_1 + \bar{j}\,\Omega_2 + \bar{k}\,\Omega_3, \qquad (6)$$

shown in Figure 1.

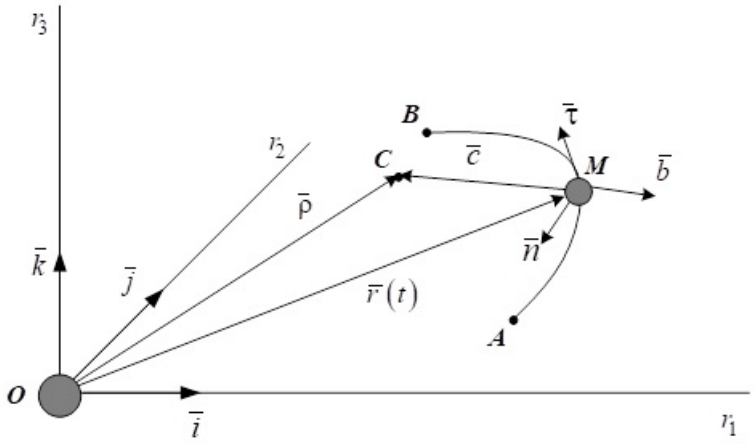


Fig. 1. Reference systems, hodograph, natural trihedral

SOLUTION METHOD

Following ref. [22], a fixed-point *C* is considered in a relative frame of reference $M\bar{\tau}\bar{n}\bar{b}$ associated with a natural trihedral. Point *C* is corresponded to the vector $MC = \bar{c}$, those:

$$\bar{c} = \bar{\tau}c_\tau + \bar{n}c_n + \bar{b}c_b, \qquad (7)$$

where c_τ, c_n, c_b – const.

In the stationary frame of reference $O\bar{i}\bar{j}\bar{k}$ point *C* is corresponded to the radius-vector $OC = \bar{\rho}$. Taking this into account, the following vector dependence of point *C* with the hodograph of programmed motion (Fig. 1) is established as follows:

$$\bar{\rho} = \bar{r} + \bar{c}. \qquad (8)$$

Using directly time differentiating of this vector equality, it can be obtained:

$$\frac{d\bar{\rho}}{dt} = \frac{d\bar{r}}{dt} + \frac{d\bar{c}}{dt}. \qquad (9)$$

It's obvious that:

$$\frac{d\bar{\rho}}{dt} = \frac{d\bar{r}}{dt} + \bar{\Omega}\times\bar{c}, \qquad (10)$$

where $\bar{\Omega}$ – angular velocity of relative coordinate system rotation, those:

$$\frac{d\bar{c}}{dt} = \bar{\Omega}\times\bar{c}.$$

Applying vector-matrix correspondence [17], it can be got:

$$\left\| 0 \quad \frac{d\bar{\tau}}{dt} \quad \frac{d\bar{n}}{dt} \quad \frac{d\bar{b}}{dt} \right\| c = \frac{1}{2} \left\| 0 \quad \bar{\tau} \quad \bar{n} \quad \bar{b} \right\| \left(\Omega_0 + {}^t\Omega_0 \right) c \quad , \quad (11)$$

or

$$\left[\left\| 0 \quad \frac{d\bar{\tau}}{dt} \quad \frac{d\bar{n}}{dt} \quad \frac{d\bar{b}}{dt} \right\| - \frac{1}{2} \left\| 0 \quad \bar{\tau} \quad \bar{n} \quad \bar{b} \right\| \left(\Omega_0 + {}^t\Omega_0 \right) \right] c = 0 \quad ,$$

where quaternion matrices:

$$\Omega_0 = \left\| \begin{matrix} 0 & \Omega_\tau & \Omega_n & \Omega_b \\ -\Omega_\tau & 0 & -\Omega_b & \Omega_n \\ -\Omega_n & \Omega_b & 0 & -\Omega_\tau \\ -\Omega_b & -\Omega_n & \Omega_\tau & 0 \end{matrix} \right\|, \quad {}^t\Omega_0 = \left\| \begin{matrix} 0 & -\Omega_\tau & -\Omega_n & -\Omega_b \\ \Omega_\tau & 0 & -\Omega_b & \Omega_n \\ \Omega_n & \Omega_b & 0 & -\Omega_\tau \\ \Omega_b & -\Omega_n & \Omega_\tau & 0 \end{matrix} \right\|, \quad c = \left\| \begin{matrix} 0 \\ c_\tau \\ c_n \\ c_b \end{matrix} \right\|.$$

Since the quaternion-column *c* is constant and arbitrary, follows:

$$\left\| 0 \quad \frac{d\bar{\tau}}{dt} \quad \frac{d\bar{n}}{dt} \quad \frac{d\bar{b}}{dt} \right\| = \frac{1}{2} \left\| 0 \quad \bar{\tau} \quad \bar{n} \quad \bar{b} \right\| \left(\Omega_0 + {}^t\Omega_0 \right) . \quad (12)$$

Transpose the vector-matrix equality, it can be obtained:

$$\left\| \begin{matrix} 0 \\ \frac{d\bar{\tau}}{dt} \\ \frac{d\bar{n}}{dt} \\ \frac{d\bar{b}}{dt} \end{matrix} \right\| = \frac{1}{2} \left({}^t\Omega_0^t + {}^t\Omega_0 \right) \left\| \begin{matrix} 0 \\ \bar{\tau} \\ \bar{n} \\ \bar{b} \end{matrix} \right\| , \quad (13)$$

or in expanded record:

$$\left\| \begin{matrix} 0 \\ \dot{\bar{\tau}} \\ \dot{\bar{n}} \\ \dot{\bar{b}} \end{matrix} \right\| = \left\| \begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \Omega_b & -\Omega_n \\ 0 & -\Omega_b & 0 & \Omega_\tau \\ 0 & \Omega_n & -\Omega_\tau & 0 \end{matrix} \right\| \left\| \begin{matrix} 0 \\ \bar{\tau} \\ \bar{n} \\ \bar{b} \end{matrix} \right\| . \quad (14)$$

Introducing the intermediate argument *s* – the length of the arc, it can be also got:

$$\dot{s} \left\| \begin{matrix} 0 \\ \frac{d\bar{\tau}}{ds} \\ \frac{d\bar{n}}{ds} \\ \frac{d\bar{b}}{ds} \end{matrix} \right\| = \left\| \begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \Omega_b & -\Omega_n \\ 0 & -\Omega_b & 0 & \Omega_\tau \\ 0 & \Omega_n & -\Omega_\tau & 0 \end{matrix} \right\| \left\| \begin{matrix} 0 \\ \bar{\tau} \\ \bar{n} \\ \bar{b} \end{matrix} \right\| . \quad (15)$$

Whence follow the vector equations connecting the sought-for components of the angular velocity of rotation of the natural trihedral of a given spiral-helix trajectory in its own relative axes Ω_τ, Ω_n, Ω_b:

$$\dot{s}\frac{d\bar{\tau}}{ds} = \Omega_b\bar{n} - \Omega_n\bar{b},$$
$$\dot{s}\frac{d\bar{n}}{ds} = -\Omega_b\bar{\tau} + \Omega_\tau\bar{b},$$
$$\dot{s}\frac{d\bar{b}}{ds} = \Omega_n\bar{\tau} - \Omega_\tau\bar{n}. \quad . \qquad (16)$$

To solve these equations, there will be used the following obvious vector formulas:

$$\frac{d\bar{\tau}}{ds} = K\cdot\bar{n}, \qquad K^2 = \frac{d\bar{\tau}}{ds}\cdot\frac{d\bar{\tau}}{ds}, \qquad K^2 = -\frac{d^2\bar{\tau}}{ds^2}\cdot\bar{\tau},$$
$$\frac{dK}{ds} = \frac{d^2\bar{\tau}}{ds^2}\cdot\bar{n}, \quad \bar{n}\times\frac{d\bar{n}}{ds} = \frac{1}{K^2}\left(\frac{d\bar{\tau}}{ds}\times\frac{d^2\bar{\tau}}{ds^2}\right), \qquad K = \frac{d\bar{\tau}}{ds}\cdot\bar{n},$$

and also

$$\bar{\tau}\cdot\frac{d\bar{\tau}}{ds} = 0, \qquad \bar{\tau}\cdot\frac{d\bar{n}}{ds} = -\bar{n}\cdot\frac{d\bar{\tau}}{ds}, \quad \bar{\tau}\cdot\frac{d\bar{b}}{ds} = -\bar{b}\cdot\frac{d\bar{\tau}}{ds},$$
$$\bar{n}\cdot\frac{d\bar{\tau}}{ds} = -\bar{\tau}\cdot\frac{d\bar{n}}{ds}, \qquad \bar{n}\cdot\frac{d\bar{n}}{ds} = 0, \qquad \bar{n}\cdot\frac{d\bar{b}}{ds} = -\bar{b}\cdot\frac{d\bar{n}}{ds},$$
$$\bar{b}\cdot\frac{d\bar{\tau}}{ds} = -\bar{\tau}\cdot\frac{d\bar{b}}{ds}, \quad \bar{b}\cdot\frac{d\bar{n}}{ds} = -\bar{n}\cdot\frac{d\bar{b}}{ds}, \qquad \bar{b}\cdot\frac{d\bar{b}}{ds} = 0$$

and

$$\bar{\tau} = \bar{n}\times\bar{b}, \quad \bar{n} = \bar{b}\times\bar{\tau}, \quad \bar{b} = \bar{\tau}\times\bar{n}.$$

Reduce the first vector equation (16) to the coordinate form and, accordingly, obtain:

$$\begin{array}{ll}
\overline{\tau}: & \left|\begin{array}{c} \dot{s}\overline{\tau}\cdot\dfrac{d\overline{\tau}}{ds}=\Omega_b\overline{\tau}\cdot\overline{n}-\Omega_n\overline{\tau}\cdot\overline{b}, \\ \overline{\tau}\cdot\dfrac{d\overline{\tau}}{ds}=0 \end{array}\right. \\
\overline{n}: & \left|\begin{array}{c} \dot{s}\overline{n}\cdot\dfrac{d\overline{\tau}}{ds}=\Omega_b\overline{n}\cdot\overline{n}-\Omega_n\overline{n}\cdot\overline{b}, \\ \Omega_b=\dot{s}\overline{n}\cdot\dfrac{d\overline{\tau}}{ds}, \quad \text{where} \quad \dfrac{d\overline{\tau}}{ds}=\overline{n}K, \quad \text{and} \\ \Omega_b=\dot{s}\overline{n}\cdot\overline{n}\cdot K, \quad \text{those} \\ \Omega_b=\dot{s}\cdot K \end{array}\right. \\
\overline{b}: & \left|\begin{array}{c} \dot{s}\overline{b}\cdot\dfrac{d\overline{\tau}}{ds}=\Omega_b\overline{b}\cdot\overline{n}-\Omega_n\overline{b}\cdot\overline{b}, \\ \Omega_n=-\dot{s}\overline{b}\cdot\dfrac{d\overline{\tau}}{ds}, \\ \Omega_n=-\dot{s}\overline{b}\cdot\overline{n}\cdot K, \quad \text{those} \\ \Omega_n=0 \end{array}\right. .
\end{array} \qquad (17)$$

For the second vector equation, it can be written as:

$$\begin{array}{ll}
\overline{\tau}: & \left|\begin{array}{c} \dot{s}\overline{\tau}\cdot\dfrac{d\overline{n}}{ds}=-\Omega_b\overline{\tau}\cdot\overline{\tau}+\Omega_\tau\overline{\tau}\cdot\overline{b}, \\ \Omega_b=-\dot{s}\overline{\tau}\cdot\dfrac{d\overline{n}}{ds}, \quad \text{where} \quad \overline{\tau}\cdot\dfrac{d\overline{n}}{ds}=-\overline{n}\cdot\dfrac{d\overline{\tau}}{ds}, \quad \text{and} \\ \Omega_b=\dot{s}\overline{n}\cdot\dfrac{d\overline{\tau}}{ds}, \\ \Omega_b=\dot{s}\overline{n}\cdot\overline{n}\cdot K \quad \text{those} \quad \Omega_b=\dot{s}\cdot K \end{array}\right. \\
\overline{n}: & \left|\begin{array}{c} \dot{s}\overline{n}\cdot\dfrac{d\overline{n}}{ds}=-\Omega_b\overline{n}\cdot\overline{\tau}+\Omega_\tau\overline{n}\cdot\overline{b}, \\ \overline{n}\cdot\dfrac{d\overline{n}}{ds}=0 \end{array}\right. \\
\overline{b}: & \left|\begin{array}{c} \dot{s}\overline{b}\cdot\dfrac{d\overline{n}}{ds}=-\Omega_b\overline{b}\cdot\overline{\tau}+\Omega_\tau\overline{b}\cdot\overline{b}, \\ \Omega_\tau=\dot{s}\overline{b}\cdot\dfrac{d\overline{n}}{ds}=\dot{s}\left(\overline{\tau}\times\overline{n}\right)\cdot\dfrac{d\overline{n}}{ds}=\dot{s}\cdot\overline{\tau}\cdot\left(\overline{n}\times\dfrac{d\overline{n}}{ds}\right), \\ \Omega_\tau=\dot{s}\dfrac{1}{K^2}\cdot\overline{\tau}\cdot\left(\dfrac{d\overline{\tau}}{ds}\times\dfrac{d^2\overline{\tau}}{ds^2}\right), \quad \text{those} \\ \Omega_\tau=\dot{s}\chi, \quad \text{where} \quad \chi=\dfrac{1}{K^2}\cdot\overline{\tau}\cdot\left(\dfrac{d\overline{\tau}}{ds}\times\dfrac{d^2\overline{\tau}}{ds^2}\right) \end{array}\right.
\end{array} \qquad (18)$$

Here χ is the torsion of the program trajectory [22]. For the third vector equation it can be found in next form:

$$\begin{aligned}
\bar{\tau}: &\quad \left| \begin{array}{l} \dot{s}\bar{\tau}\cdot\dfrac{d\bar{b}}{ds}=\Omega_n\bar{\tau}\cdot\bar{\tau}-\Omega_\tau\bar{\tau}\cdot\bar{n}, \\ \Omega_n=\dot{s}\bar{\tau}\cdot\dfrac{d\bar{b}}{ds}, \quad \text{where} \quad \bar{\tau}\cdot\dfrac{d\bar{b}}{ds}=-\bar{b}\cdot\dfrac{d\bar{\tau}}{ds}, \quad \text{and} \\ \Omega_n=-\dot{s}\bar{b}\cdot\dfrac{d\bar{\tau}}{ds}=-\dot{s}\bar{b}\cdot\bar{n}\cdot K=0 \end{array} \right. \\
\bar{n}: &\quad \left| \begin{array}{l} \dot{s}\bar{n}\cdot\dfrac{d\bar{b}}{ds}=\Omega_n\bar{n}\cdot\bar{\tau}-\Omega_\tau\bar{n}\cdot\bar{n}, \\ \Omega_\tau=-\dot{s}\bar{n}\cdot\dfrac{d\bar{b}}{ds}=\dot{s}\bar{b}\cdot\dfrac{d\bar{n}}{ds}=\dot{s}\left(\bar{\tau}\times\bar{n}\right)\cdot\dfrac{d\bar{n}}{ds}=\dot{s}\bar{\tau}\cdot\left(\bar{n}\times\dfrac{d\bar{n}}{ds}\right), \\ \Omega_\tau=\dot{s}\chi, \quad \text{where} \quad \chi=\bar{\tau}\cdot\left(\bar{n}\times\dfrac{d\bar{n}}{ds}\right) \end{array} \right. \\
\bar{b}: &\quad \left| \begin{array}{l} \dot{s}\bar{b}\cdot\dfrac{d\bar{b}}{ds}=\Omega_n\bar{b}\cdot\bar{\tau}-\Omega_\tau\bar{b}\cdot\bar{n}, \\ \bar{b}\cdot\dfrac{d\bar{b}}{ds}=0. \end{array} \right.
\end{aligned} \qquad (19)$$

This is consistent with previously established results. In the matrix representation, the well-known Frenet-Serret [22] formulas take the ordered form:

$$\begin{Vmatrix} 0 \\ \dfrac{d\bar{\tau}}{ds} \\ \dfrac{d\bar{n}}{ds} \\ \dfrac{d\bar{b}}{ds} \end{Vmatrix} = \begin{Vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & K & 0 \\ 0 & -K & 0 & \chi \\ 0 & 0 & -\chi & 0 \end{Vmatrix} \begin{Vmatrix} 0 \\ \bar{\tau} \\ \bar{n} \\ \bar{b} \end{Vmatrix} . \qquad (20)$$

Thus, the sought vector of angular velocity in the axes of the natural trihedral is represented by the following vector-matrix form:

$$\begin{Vmatrix} 0 & \bar{\tau} & \bar{n} & \bar{b} \end{Vmatrix} \begin{Vmatrix} 0 \\ \Omega_\tau \\ \Omega_n \\ \Omega_b \end{Vmatrix} = \dot{s} \begin{Vmatrix} 0 & \bar{\tau} & \bar{n} & \bar{b} \end{Vmatrix} \begin{Vmatrix} 0 \\ \chi \\ 0 \\ K \end{Vmatrix}, \qquad (21)$$

those

$$\bar{\Omega}=\bar{\tau}\cdot\dot{s}\cdot\chi+\bar{n}\cdot 0+\bar{b}\cdot\dot{s}\cdot K . \qquad (22)$$

The components of the angular velocity vector $\Omega_1,\Omega_2,\Omega_3$ of rotation of the natural trihedral of the trajectory in a stationary frame of reference are found according to a given hodograph using the following formulas:

$$\bar{\tau}=\frac{\dot{\bar{r}}}{\left|\dot{\bar{r}}\right|},\qquad \bar{b}=\frac{\dot{\bar{r}}\times\ddot{\bar{r}}}{\left|\dot{\bar{r}}\times\ddot{\bar{r}}\right|},$$
$$K=\frac{\left|\dot{\bar{r}}\times\ddot{\bar{r}}\right|}{\left|\dot{\bar{r}}\right|^3},\quad \chi=\frac{\dot{\bar{r}}\cdot\left(\ddot{\bar{r}}\times\dddot{\bar{r}}\right)}{\left|\dot{\bar{r}}\times\ddot{\bar{r}}\right|^2}. \qquad (23)$$

where $\left|\dot{\bar{r}}\right|=\dot{s}$,

$$\left|\dot{\bar{r}}\right|^2=\dot{\bar{r}}\cdot\dot{\bar{r}},$$
$$\left|\dot{\bar{r}}\times\ddot{\bar{r}}\right|^2=\left(\dot{\bar{r}}\times\ddot{\bar{r}}\right)\cdot\left(\dot{\bar{r}}\times\ddot{\bar{r}}\right),$$

or in quaternion matrices [19]:

$$\dot{\bar{r}}\cdot\dot{\bar{r}}\sim\frac{1}{2}\left(\dot{R}_0+\dot{R}_0^t\right)\dot{r}_0,$$
$$\dot{\bar{r}}\times\ddot{\bar{r}}\sim\frac{1}{2}\left(\dot{R}_0-\dot{R}_0^t\right)\ddot{r}_0,$$
$$\dot{\bar{r}}\cdot\left(\ddot{\bar{r}}\times\dddot{\bar{r}}\right)\sim\frac{1}{2}\dot{r}_0^t\left(\ddot{R}_0-\ddot{R}_0^t\right)\dddot{r}_0,$$
$$\left(\dot{\bar{r}}\times\ddot{\bar{r}}\right)\cdot\left(\dot{\bar{r}}\times\ddot{\bar{r}}\right)\sim\frac{1}{4}\dot{r}_0^t\left(\ddot{R}_0-\ddot{R}_0^t\right)\left(\dot{R}_0-\dot{R}_0^t\right)\ddot{r}_0. \qquad (24)$$

Here:

$$\dot{R}_0=\begin{Vmatrix}0&\dot{r}_1&\dot{r}_2&\dot{r}_3\\-\dot{r}_1&0&-\dot{r}_3&\dot{r}_2\\-\dot{r}_2&\dot{r}_3&0&-\dot{r}_1\\-\dot{r}_3&-\dot{r}_2&\dot{r}_1&0\end{Vmatrix},\quad \dot{R}_0^t=\begin{Vmatrix}0&\dot{r}_1&\dot{r}_2&\dot{r}_3\\-\dot{r}_1&0&\dot{r}_3&-\dot{r}_2\\-\dot{r}_2&-\dot{r}_3&0&\dot{r}_1\\-\dot{r}_3&\dot{r}_2&-\dot{r}_1&0\end{Vmatrix}$$
$$\dot{r}_0=\begin{Vmatrix}0\\\dot{r}_1\\\dot{r}_2\\\dot{r}_3\end{Vmatrix},\quad \ddot{r}_0=\begin{Vmatrix}0\\\ddot{r}_1\\\ddot{r}_2\\\ddot{r}_3\end{Vmatrix},\quad \dddot{r}_0=\begin{Vmatrix}0\\\dddot{r}_1\\\dddot{r}_2\\\dddot{r}_3\end{Vmatrix},$$
$$\dot{r}_0^t=\begin{Vmatrix}0&\dot{r}_1&\dot{r}_2&\dot{r}_3\end{Vmatrix}.$$

Then it can be established:

$$\bar{\Omega}=\chi\dot{\bar{r}}+\frac{\dot{\bar{r}}\times\ddot{\bar{r}}}{\dot{\bar{r}}\cdot\dot{\bar{r}}}, \qquad (25)$$

or in coordinate form:

$$\bar{\Omega} = \bar{i}\left(\chi\dot{r}_1 + \frac{\begin{vmatrix}\dot{r}_2 & \dot{r}_3\\ \ddot{r}_2 & \ddot{r}_3\end{vmatrix}}{\dot{\bar{r}}\cdot\ddot{\bar{r}}}\right) + \bar{j}\left(\chi\dot{r}_2 - \frac{\begin{vmatrix}\dot{r}_1 & \dot{r}_3\\ \ddot{r}_1 & \ddot{r}_3\end{vmatrix}}{\dot{\bar{r}}\cdot\ddot{\bar{r}}}\right) + \bar{k}\left(\chi\dot{r}_3 + \frac{\begin{vmatrix}\dot{r}_1 & \dot{r}_2\\ \ddot{r}_1 & \ddot{r}_2\end{vmatrix}}{\dot{\bar{r}}\cdot\ddot{\bar{r}}}\right). \quad (26)$$

Thus, the components of the angular velocity of rotation of the natural trihedral are determined in a stationary frame of reference according to the hodograph of the programmed motion in the form:

$$\Omega_1 = \chi\dot{r}_1 + \frac{\begin{vmatrix}\dot{r}_2 & \dot{r}_3\\ \ddot{r}_2 & \ddot{r}_3\end{vmatrix}}{\dot{\bar{r}}\cdot\ddot{\bar{r}}}, \quad \Omega_2 = \chi\dot{r}_2 - \frac{\begin{vmatrix}\dot{r}_1 & \dot{r}_3\\ \ddot{r}_1 & \ddot{r}_3\end{vmatrix}}{\dot{\bar{r}}\cdot\ddot{\bar{r}}}, \quad \Omega_3 = \chi\dot{r}_3 + \frac{\begin{vmatrix}\dot{r}_1 & \dot{r}_2\\ \ddot{r}_1 & \ddot{r}_2\end{vmatrix}}{\dot{\bar{r}}\cdot\ddot{\bar{r}}}. \quad (27)$$

DISCUSSION OF RESULTS

The presented approach of using unit tangent, normal and binormal vectors, quaternion matrices (7)-(27) for analysis the kinematic parameters of natural trihedral can be used in a wide class of mechanical problems of studying the motion of a rigid body. So, to analyze the angular motion of flight object (spacecraft, aircraft, launch vehicle, missle, etc.), a certain relative reference frame connected with the center of mass of this flight object [23, 24] is introduced. Rotation relative to this coordinate system determine the angles of current orientation of flight object (Euler or Krylov angles). However, using special transform rotation matrices from one coordinate system to another based special rotative angles can cause to large errors in long-periodic predict of motion. This is explained by the fact that these angles of rotation are integrable parameters, the numerical integration of which leads to the accumulation of errors over long periods (for example, a decade duration). Based on this, the problem of analyzing the orientation and stabilization of a spacecraft over ultra-long time periods is reduced to piecewise-continuous modeling at selected intervals.

In turn, the application of quaternion matrices using the transition based on the Frenet-Serret frame, which is presented in this work, can minimize these errors. In turn, this can make it possible to increase the time intervals for studying the angular motion of the spacecraft in piecewise-continuous modeling. That's why, the correct determination of this relative reference frame connected with the center of mass is very actual in aerospace long-predictive dynamics.

CONCLUSIONS

The hodograph of the programmed motion of transport vehicle in the class of spiral-helix trajectories in a stationary frame of reference is considered. The kinematic parameters of the natural trihedral of the trajectory in vector-matrix form: the unit tangent, normal and binormal vectors; curvature; torsion; the components of the angular velocity of rotation of the natural trihedral in its own basis and in the basis of a stationary frame of references have been studied. Presented mathematical algorithms in quaternion matrices are adapted to computer calculations. The ordered, matrix results are verified by comparison with the well-known Frenet-Serret formulas.

References

Lazaryan, V. A., Dinamyka transportnyh sredstv [Vehicle dynamics], Kyiv, Naukova dumka, 1985, 528 p. (in Russian).

Blokhin, E. P., Manashkin, L. A., Dinamyka poezda [The dynamics of the train], Moscow, Transport, 1982, 222 p. (in Russian).

Lysenko, L. N., Navedenie i navigatsion ballisticheskih raket [Navigation and guidance ballistic rockets], Moscow, Bauman university publ., 2007, 762 p. (in Russian).

Igdalov, I. M., Kuchma, L. D., Polyakov, N. V., Sheptun, Yu. D., Dinamicheskoe proektirovanie raket. Zadachi dinamiki raket I ih kosmicheskih stupeney [Dynamic design of missiles. The tasks of the dynamics rockets and their cosmic stages], Dnipro, Dnipro National University publ., 2010, 264 p. (in Russian).

Kravets, V. V., Kravets, Vl. V., Burov, O. V., Reliability of Systems. Part 1. Statics of Failures. Lap Lambert Academic Publishing, Omni Scriptum GmbH&Co. KG. 2016. 93 p. ISBN: 978-3-659-83413-4.

Kravets, V. V., Kravets, Vl. V., Burov, O. V., Reliability of Systems. Part 2. Dynamics of Failures. Lap Lambert Academic Publishing, Omni Scriptum GmbH&Co. KG. 2016. 100 p. ISBN: 978-3-659-89711-5.

Kravets, V. V., Kravets, T. V. On the nonlinear dynamics of elastically interacting asymmetric rigid bodies. Int. Appl. Mech. 2006. Vol. 42. No. 1. P 110 – 114.

Kravets, V. V., Kravets, T. V., Kharchenko, A. V. Using quaternion matrices to describe the kinematics and nonlinear dynamics of an asymmetricrigid body. Int. Appl. Mech. 2009. Vol. 44. No. 2. P 223 – 232.

Kravets, V.V., Bas, K.M., Kravets T.V., Tokar, L.M., Dynamic design of ground transport with the help of computation experiment, MMSE Journal, 2015, 1, p.p. 105–111. ISSN 2412-5954, Open access www.mmse.xyz, DOI 10.13140/RG.2.1.2466.6643.

Kravets, V. V., Kravets, Vl. V., Fedoriachenko, S. A., On Application of the Ground Effect For Highspeed Surface Vehicles. Mechanics, Materials Science&Engineering. 2016. No. 4. P. 82–87. DOI:10.13140/RG.2.1.1034.5365.

Kravets, V.V., Bas, K.M., Kravets T.V., Zubariev, M.S., Tokar, L.A., Kinetostatics of wheel vehicle in the category of spiral-screw routes, MMSE Journal, 2016, 5, p.p. 87-89. Open access www.mmse.xyz, DOI: 10.13140/RG.2.1.10109.3921.

Beshta, O., Kravets, V., Bas, K., Kravets, T., Tokar, L. Control of tandem-type two-wheel vehicle at various notion modes along spatial curved lay of line. Power Engineering, Control and Information Technologies in Geotechnical Systems, London, Taylor and Francis Group, 2015, ISBN 978-1-138-02804-3, p. 27-32. DOI: 10.1201/b18475-6.

Pivnyak G. G., Sakhno V. P., Kravets V. V., Bas K. M. Method for determining high-speed vehicle contact forces of the ground transport. Naukovyi Visnyk NHU. 2019. No. 1, P. 55 – 61. DOI: 10.29202/nvngu/2019-1/8.

Kravets V., Sakhno, V., Bas, K., Kravets, Vl., Program spatial moment of highspeed vehicles, CMSME 2018, IOP Conf., Series: Material science and engineering, IOP Publ., 2018, p. 383. DOI: 10.1088/1757-899X/383/1/012032.

Kravets, V.V., Kravets, Vl.V., Artemchuk, V.V., The Vehicle Controlling Near the Screening Surface Using Thrust Vector Deflection of the Electric Motor with Gimbal Mounted Propeller, Mechanics, Materials Science & Engineering, 2017, 12, p.p. 117-122. DOI:10.2412/mmse.2.35.544.

Kravets V.V., Kravets Vl.V., Artemchuk V.V. Control of following Force in Gimbal Suspension by Programmed Transfer of Vehicle in the Class of Spiral-Screw Trajectories. TMLAI, vol. 8, no. 6. December, 2020, P. 1–15. DOI: 10.14738/tmlai.86.8743

Kravets, V., Kravets, T., Burov, O., Monomial (1, 0, -1)-matrices-(4x4). Part 1. Application to the transfer in space. Lap Lambert Academic Publishing, Omni Scriptum GmbH&Co. KG., 2016, 137 p. ISBN: 978-3-330-01784-9.

Kravets, V., Kravets, T., Burov, O., Monomial (1, 0, -1)-matrices-(4x4). Part 2. Application to the rotation in space. Lap Lambert Academic Publishing, Omni Scriptum GmbH&Co. KG., 2017, 82 p. ISBN: 978-3-330-34185-2.

Kravets, V., Kravets, T., Burov, O., Monomial matrices. Part 3. Application to the programmed transfer and rotation. Lap Lambert Academic Publishing, Omni Scriptum Publishing Group., 2019, 124 p. ISBN: 978-620-0-08354-8.

Sakhno V. P., Kravets V. V., Bas K. M., Krivda V. V. Orientation of natural trihedral of the spiral-helix supporting trajectory of spatial vehicle movement. Naukovyi Visnyk NHU, 2018, No. 3, P. 50 – 56. DOI: 10.29202/nvngu/2018-3/15

Kravets, V.V., Kravets, T.V., Bas, K.M., Tokar, L.M., Mathematical model of a path hodograph of surface transport, Transport problems, Dnipro, 2014, p. 830–841..

Lur'e, A.I., Analytical mechanics, Springer Science & Business Media, 2002, 824 p.

Alpatov A.P., Khoroshylov S.V., Maslova A.I. Contactless de-orbiting of space debris by the ion beam. Dynamics and control. Kyiv: Akademperiodyka, 2019. 170 p.

Alpatov A., Khoroshylov S., Lapkhanov E. Synthesizing an algorithm to control the angular motion of spacecraft equipped with an aeromagnetic deorbiting system. Eastern-European Journal of Enterprise Technologies. 2020. Vol. 1. Iss. 5(103). Pp. 37 – 46. DOI: 10.15587/1729-4061.2020.192813