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Analytical Modeling of the Dynamic System of the Fourth Order

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ABSTRACT

A canonical mathematical model of a fourth-order dynamical system in the form of *A.M. Letov.* The analytical modeling methods are based on the algebraic concept and the principle of symmetry. The symmetry principle is realized on the set of four indices of the roots of the characteristic equation and the set of four indices of the phase coordinates of the dynamic system. The problem of the quality of dynamic processes in time is reduced to the algebraic problem of distribution of four roots in the complex plane. An analogy is established in the procedure for transforming the characteristic determinant to a polynomial and elementary symmetric polynomials of four roots. On the basis of the theory of residues, a new form of analytical representation of data in time is obtained in the form of ordered determinants with respect to the indices of four roots and indices of four coordinates. General provisions are illustrated by a stochastic dynamical system in the form of an asymmetric Markov chain with four states and continuous time, which is described by the fourth-order Kolmogorov equations.

Keywords: Mathematical model, symmetric polynomials, central symmetry, phase coordinates.

INTRODUCTION

Mathematical modeling as a tool for solving problems of analysis and synthesis of deterministic or stochastic dynamic systems is developing in a complex way, combining the improvement of mathematical methods and computer technologies.

Analytical modeling is an important stage in the dynamic design of technical systems, which precedes computational and full-scale experiments [1-4].

The central place in analytical modeling is occupied by a correctly composed mathematical model that adequately describes deterministic or stochastic processes in differential form,

relying on fundamental laws and heuristic postulates. Various mathematical models of dynamical systems are represented by the canonical, matrix form [5].

Analytical modeling is carried out on the basis of classical mathematical methods for solving systems of linear differential equations, operational calculus, theory of residues [6, 7].
Analytical modeling of a second-order linear dynamical system is reduced to the algebraic problem of determining the roots of a quadratic characteristic equation and the subsequent analysis of differential equations [8-11].

With an increase in the order of the dynamical system, analytical modeling encounters the problem of solvability of the characteristic equations of high degrees [5].

In this paper, using the example of a fourth-order dynamical system, we show the possibility of overcoming this fundamental limitation by using the proposed special solution of the complete algebraic equation of the fourth degree [12].

Approbation of the method is illustrated by considering an asymmetric Markov chain with four states and continuous time, which is described by the Kolmogorov equations [13].

The proposed mathematical models make it possible to expand the range of problems of analysis and synthesis of dynamical systems, simulated analytically.

SETTING UP A PROBLEM

A wide class of problems in the dynamics of both deterministic and stochastic design schemes of objects is reduced to the consideration of mathematical models in the form of systems of linear, homogeneous differential equations with constant coefficients for a given state of the object at the initial moment. [1,4,9].

As an illustration, a mathematical model of the fourth order in canonical form is considered [5]:

$$\begin{aligned} \dot{x}_1 &= a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4, \\ \dot{x}_2 &= a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4, \\ \dot{x}_3 &= a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + a_{34}x_4, \\ \dot{x}_4 &= a_{41}x_1 + a_{42}x_2 + a_{43}x_3 + a_{44}x_4. \end{aligned} \qquad (1)$$

We are looking for the possibility of analytical representation of dynamic processes in an ordered, universal form, convenient for solving engineering problems of analysis and synthesis.

ANALYTICAL SOLUTION

The analytical solution of the original system of differential equations is reduced to the algebraic problem of determining the roots of the characteristic equation [7]:

$$\begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22}-\lambda & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33}-\lambda & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44}-\lambda \end{vmatrix} = 0. \qquad (2)$$

Here the characteristic determinant is transformed to a fourth degree polynomial:

$$\sum_{i=0}^{4} a_{4-i}^{4}\,\lambda^{4-i} = 0, \quad (i = 0, 1, 2, 3, 4). \tag{3}$$

According to *Vieta's formulas*, the coefficients of an algebraic equation of the fourth degree are related to the four roots of the characteristic equation λ_1, λ_2, λ_3, λ_4 by the following symmetric polynomials [5]:

$$i = 4: \quad a_0^4 = (-1)^0 \prod_{j=1}^{4} \lambda_j\,, \tag{4}$$

$$i = 3: \quad a_1^4 = (-1)^1 \sum_{\substack{j,k,s=1 \\ (j<k<s)}}^{4} \lambda_j \lambda_k \lambda_s, \tag{5}$$

$$i = 2: \quad a_2^4 = (-1)^2 \sum_{\substack{j,k=1 \\ (j<k)}}^{4} \lambda_j \lambda_k\,, \tag{6}$$

$$i = 1: \quad a_3^4 = (-1)^3 \sum_{j=1}^{4} \lambda_j\,, \tag{7}$$

$$i = 0: \quad a_4^4 = (-1)^4\,. \tag{8}$$

It is easy to show the similarity of the structures of the formulas for the coefficients of the characteristic equation of the fourth degree, expressed by symmetric polynomials of the roots and symmetric polynomials of special determinants of the fourth order, constructed from the columns of the original characteristic determinant as follows:

$$i = 0: \quad a_4^4 = \begin{vmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix}, \tag{9}$$

$$i=1:\quad a_3^4=\sum_{j=1}^{4}\begin{vmatrix} -1 & 0\cdots a_{1j}\cdots 0\\ 0 & -1\cdots a_{2j}\cdots 0\\ \vdots & \vdots\ \ \vdots\ \ \vdots\\ 0 & 0\cdots a_{jj}\cdots 0\\ \vdots & \vdots\ \ \vdots\ \ \vdots\\ 0 & 0\cdots a_{4j}\cdots -1 \end{vmatrix}, \tag{10}$$

$$i=2:\quad a_2^4=\sum_{\substack{j,k=1\\(j<k)}}^{4}\begin{vmatrix} -1 & \cdots a_{1j}\cdots a_{1k}\cdots 0\\ \vdots & \vdots\ \ \vdots\ \ \vdots\\ 0\cdots & a_{jj}\cdots a_{jk}\cdots 0\\ \vdots & \vdots\ \ \vdots\ \ \vdots\\ 0 & a_{kj}\cdots a_{kk}\cdots 0\\ \vdots & \vdots\ \ \vdots\ \ \vdots\\ 0\cdots & a_{4j}\cdots a_{4k}\cdots -1 \end{vmatrix}, \tag{11}$$

$$i=3:\quad a_1^4=\sum_{\substack{j,k,s=1\\(j<k<s)}}^{4}\begin{vmatrix} -1 & \cdots a_{1j}\cdots a_{1k}\cdots a_{1s}\cdots 0\\ \vdots & \vdots\ \ \vdots\ \ \vdots\ \ \vdots\\ 0\cdots & a_{jj}\cdots a_{jk}\cdots a_{js}\cdots 0\\ \vdots & \vdots\ \ \vdots\ \ \vdots\ \ \vdots\\ 0 & a_{kj}\cdots a_{kk}\cdots a_{ks}\cdots 0\\ \vdots & \vdots\ \ \vdots\ \ \vdots\ \ \vdots\\ 0 & a_{sj}\cdots a_{sk}\cdots a_{ss}\cdots 0\\ \vdots & \vdots\ \ \vdots\ \ \vdots\ \ \vdots\\ 0\cdots & a_{4j}\cdots a_{4k}\cdots a_{4s}\cdots -1 \end{vmatrix}, \tag{12}$$

$$i=4:\quad a_0^4=\begin{vmatrix} a_{11} & a_{12} & a_{12} & a_{14}\\ a_{21} & a_{22} & a_{23} & a_{24}\\ a_{31} & a_{32} & a_{33} & a_{34}\\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix}. \tag{13}$$

A special solution to an algebraic equation of the fourth degree

In the scientific works of *Niels Henrik Abel* and *Évariste Galois* it is proved that the roots of the complete algebraic equation of the fifth and higher degrees in radicals are not expressed and constitute a more complex problem than radicals [5].

The general solutions found for equations of the third degree (*Cardano's formula*) and fourth degree (*Ferrari method*) are complex, cumbersome, and have not received wide application in practical problems. Solutions of particular types of algebraic equations of the fourth degree turned out to be more useful for applied problems:

- biquadratic equation when the coefficients satisfy the conditions $a_1^4=0,\ a_3^4=0$;
- reciprocal equation when the coefficients satisfy the conditions $a_0^4=a_4^4,\ a_1^4=a_3^4$.

A new special solution of the complete algebraic equation of the fourth degree is proposed [12]:

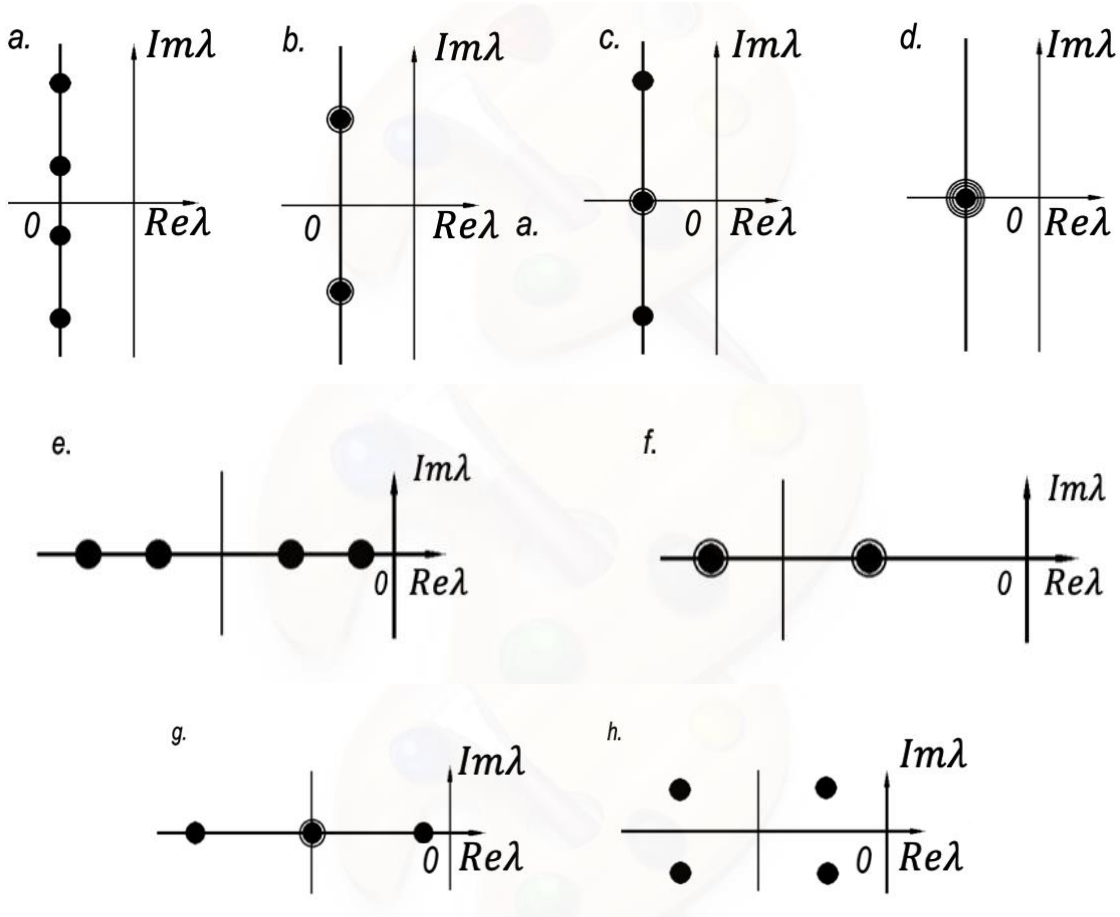
$$\lambda_{1,2,3,4} = -\frac{a_3^4}{4} \pm \frac{1}{\sqrt{2}} \sqrt{\left[\frac{3}{2}\left(\frac{a_3^4}{2}\right)^2 - a_2^4\right] \pm \sqrt{\left[\left(\frac{a_3^4}{2}\right)^2 - a_2^4\right]^2 - 4a_0^4}}. \quad (14)$$

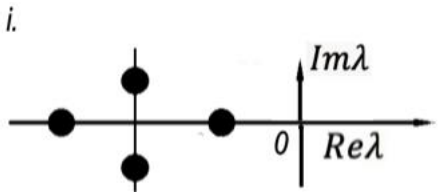
The equation is valid when the coefficients $a_1^4,\ a_2^4,\ a_3^4$ are related by the condition:

$$a_1^4 + \frac{a_3^4}{2}\left[\left(\frac{a_3^4}{2}\right)^2 - a_2^4\right] = 0. \quad (15)$$

Keeping the simplicity of the mathematical model, the proposed solution contains as a special case the known solution of the biquadratic equation. Indeed, for a biquadratic equation, the established condition is satisfied identically, and the proposed special solution takes the known form.

We also note that the distribution of roots in the complex plane, determined by this solution and the condition, has a qualitatively important property: central symmetry according to the schemes shown in Pict. 1.





Picture 1. Schemes of distribution of the roots of an algebraic equation of the fourth degree of a special form

Summary. A structurally simple, observable formula for the roots of a complete algebraic equation of the fourth degree of a special form is proposed, which makes it possible to expand the range of problems that can be analytically solvable.

Analytical solution of the canonical system of fourth order differential equations

The analytical solution of the original canonical system of differential equations is presented using the theory of residues [6]; it is presented in the following ordered form, harmonious with respect to the indices of the state variables $j = 1, 2, 3, 4$ and the roots of the characteristic equation $k = 1, 2, 3, 4$:

$$x_j = \sum_{k=1}^{4} \frac{\Delta_j(\lambda_k)}{\prod_{\substack{s=1 \\ (s \neq k)}}^{4} (\lambda_k - \lambda_s)} e^{\lambda_k t}, \ \ (j = 1, 2, 3, 4). \tag{16}$$

Here $\Delta_j(\lambda_k)$ – is an ordered determinant formed by the index of the state variable of the dynamic system by replacing in the characteristic determinant $\Delta(\lambda_k)$ the column corresponding to the variable number with the column vector of the state of the dynamic system at the initial moment of time, that is:

$$\Delta_1(\lambda_k) = \begin{vmatrix} -x_1(0) & a_{12} & a_{13} & a_{14} \\ -x_2(0) & a_{22} - \lambda_k & a_{23} & a_{24} \\ -x_3(0) & a_{32} & a_{33} - \lambda_k & a_{34} \\ -x_4(0) & a_{42} & a_{43} & a_{44} - \lambda_k \end{vmatrix}, \tag{17}$$

$$\Delta_2(\lambda_k) = \begin{vmatrix} a_{11} - \lambda_k & -x_1(0) & a_{13} & a_{14} \\ a_{21} & -x_2(0) & a_{23} & a_{24} \\ a_{31} & -x_3(0) & a_{33} - \lambda_k & a_{34} \\ a_{41} & -x_4(0) & a_{43} & a_{44} - \lambda_k \end{vmatrix}, \tag{18}$$

$$\Delta_3(\lambda_k) = \begin{vmatrix} a_{11} - \lambda_k & a_{12} & -x_1(0) & a_{14} \\ a_{21} & a_{22} - \lambda_k & -x_2(0) & a_{24} \\ a_{31} & a_{32} & -x_3(0) & a_{34} \\ a_{41} & a_{42} & -x_4(0) & a_{44} - \lambda_k \end{vmatrix}, \tag{19}$$

$$\Delta_4(\lambda_k) = \begin{vmatrix} a_{11}-\lambda_k & a_{12} & a_{13} & -x_1(0) \\ a_{21} & a_{22}-\lambda_k & a_{23} & -x_2(0) \\ a_{31} & a_{32} & a_{33}-\lambda_k & -x_3(0) \\ a_{41} & a_{42} & a_{43} & -x_4(0) \end{vmatrix}. \quad (20)$$

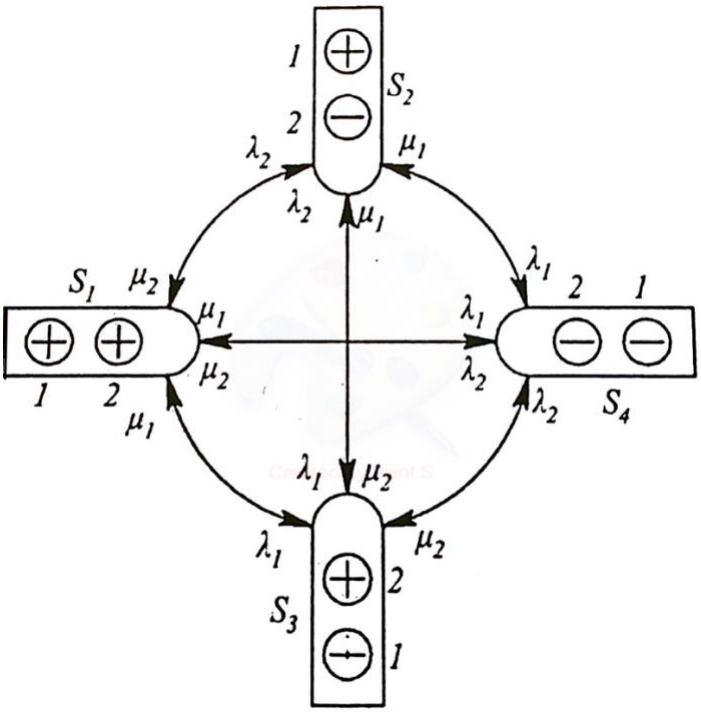
Summary. Simulation of the dynamic processes of the fourth-order canonical system can be carried out analytically, using the proposed solution of differential equations in harmonious notation, conservative with respect to the indices of the roots and phase variables.

APPROBATION OF THE METHOD

Approbation of the stated approach to analytical modeling of a stochastic dynamical system is carried out on the example of an asymmetric Markov chain with four constants and continuous time [13].

State graph

Markov random processes are represented by an asymmetric state graph (Pict. 2).



Picture 2. Asymmetric Markov chain graph with four constants

Here $S_i, (i = 1, 2, 3, 4)$ are states technical system involving two independent subsystems. The subsystems in two conditions: $\oplus$ - operational condition and $\ominus$ - inoperable one.

Intensity of failure flows λ_1, λ_2 and recovery flows μ_1, μ_2 of the two subsystems is set as given and constant. It is required to determine state probability of interactive technical system $P_i(t), (i = 1, 2, 3, 4)$ under certain condition of the system in start time $P_i(0)$.

Kolmogorov equation

Kolmogorov equation are set up in accordance with set state graph taking following symmetrical matrix form:

$$\frac{d}{dt}\begin{Vmatrix} P_1(t) \\ P_2(t) \\ P_3(t) \\ P_4(t) \end{Vmatrix} = \begin{Vmatrix} -(\lambda_1+\lambda_2) & \mu_2 & \mu_1 & 0 \\ \lambda_2 & -(\lambda_1+\mu_2) & 0 & \mu_1 \\ \lambda_1 & 0 & -(\mu_1+\lambda_2) & \mu_2 \\ 0 & \lambda_1 & \lambda_2 & -(\mu_1+\mu_2) \end{Vmatrix} \begin{Vmatrix} P_1(t) \\ P_2(t) \\ P_3(t) \\ P_4(t) \end{Vmatrix}, \quad (21)$$

where the matrix of failures and recoveries is ordered, i.e. when it is asymmetrical to the main diagonal and symmetrical to secondary diagonal, then is has central symmetry. Subsequently, characteristic determinant are:

$$\Delta(\nu) = \begin{vmatrix} -(\lambda_1+\lambda_2)-\nu & \mu_2 & \mu_1 & 0 \\ \lambda_2 & -(\lambda_1+\mu_2)-\nu & 0 & \mu_1 \\ \lambda_1 & 0 & -(\mu_1+\lambda_2)-\nu & \mu_2 \\ 0 & \lambda_1 & \lambda_2 & -(\mu_1+\mu_2)-\nu \end{vmatrix}. \quad (22)$$

Characteristic equation

Due to the ordered structure, the determinant of the characteristic equation of the fourth degree,

$$\nu^4 + a_3\nu^3 + a_2\nu^2 + a_1\nu + a_0 = 0 \quad (23)$$

is conditioned by the following coefficients:

$$a_3 = 2[(\lambda_1+\mu_1)+(\lambda_2+\mu_2)]; \quad (24)$$

$$a_2 = [(\lambda_1+\mu_1)+(\lambda_2+\mu_2)]^2 + (\lambda_1+\mu_1)(\lambda_2+\mu_2); \quad (25)$$

$$a_1 = [(\lambda_1+\mu_1)+(\lambda_2+\mu_2)](\lambda_1+\mu_1)(\lambda_2+\mu_2); \quad (26)$$

$$a_0 = 0. \quad (27)$$

It is easy to show that these coefficients identically satisfy the condition:

$$a_1 + \frac{a_3}{2}\left[\left(\frac{a_3}{2}\right)^2 - a_2\right] = 0. \quad (28)$$

Analytical solution of the characteristic equation

Consequently, the roots of the characteristic equation are found analytically using the proposed formula [12].

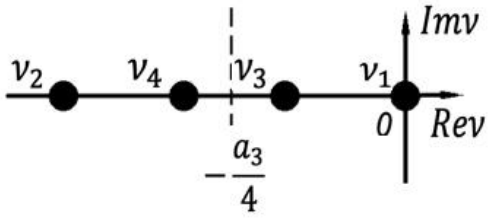
In the particular case, when $a_0 = 0$, we get:

$$\nu_1 = 0, \qquad \nu_2 = -\frac{a_3}{2},$$
$$\nu_3 = -\frac{a_3}{4} - \sqrt{\frac{5}{4}\left(\frac{a_3}{2}\right)^2 - a_2}, \qquad \nu_4 = -\frac{a_3}{4} + \sqrt{\frac{5}{4}\left(\frac{a_3}{2}\right)^2 - a_2}. \tag{29}$$

Respectively, expressing roots of characteristic equation in terms of failures and recoveries of the two subsystems, we obtain:

$$\nu_1 = 0; \quad \nu_2 = -(\lambda_1 + \mu_1) - (\lambda_2 + \mu_2);$$
$$\nu_3 = -(\lambda_1 + \mu_1); \quad \nu_4 = -(\lambda_2 + \mu_2). \tag{30}$$

That is, the roots are negative, real, different and distributed in the complex plane symmetrically (Pict. 3) about the center $Re\nu = -\frac{a_3}{4}$ according to the Pict.1.*e*:



Picture 3. Root distribution in an asymmetric Markov chain with four states

Analytical solution of the Kolmogorov equations

Symbolize analytical solution of initial matrix fourth-order Kolmogorov equation in a normal form based on structurally simple determinants:

$$P_i(t) = \sum_{k=1}^{4} \frac{\Delta_i(\nu_k)}{\prod_{\substack{s=1 \\ (k\neq s)}}^{4} (\nu_k - \nu_s)} e^{\nu_k t} \qquad (i = 1,2,3,4). \tag{31}$$

Here

$$\Delta_1(\nu_k) = \begin{vmatrix} -P_1(0) & \mu_2 & \mu_1 & 0 \\ -P_2(0) & -(\lambda_1+\mu_2)-\nu_k & 0 & \mu_1 \\ -P_3(0) & 0 & -(\mu_1+\lambda_2)-\nu_k & \mu_2 \\ -P_4(0) & \lambda_1 & \lambda_2 & -(\mu_1+\mu_2)-\nu_k \end{vmatrix}, \tag{32}$$

$$\Delta_2(\nu_k) = \begin{vmatrix} -(\lambda_1+\lambda_2)-\nu_k & -P_1(0) & \mu_1 & 0 \\ \lambda_2 & -P_2(0) & 0 & \mu_1 \\ \lambda_1 & -P_3(0) & -(\mu_1+\lambda_2)-\nu_k & \mu_2 \\ 0 & -P_4(0) & \lambda_2 & -(\mu_1+\mu_2)-\nu_k \end{vmatrix}, \tag{33}$$

$$\Delta_3(\nu_k) = \begin{vmatrix} -(\lambda_1+\lambda_2)-\nu_k & \mu_2 & -P_1(0) & 0 \\ \lambda_2 & -(\lambda_2+\mu_2)-\nu_k & -P_2(0) & \mu_1 \\ \lambda_1 & 0 & -P_3(0) & \mu_2 \\ 0 & \lambda_1 & -P_4(0) & -(\mu_1+\mu_2)-\nu_k \end{vmatrix}, \quad (34)$$

$$\Delta_4(\nu_k) = \begin{vmatrix} -(\lambda_1+\lambda_2)-\nu_k & \mu_2 & \mu_1 & -P_1(0) \\ \lambda_2 & -(\lambda_2+\mu_2)-\nu_k & 0 & -P_2(0) \\ \lambda_1 & 0 & -(\mu_1+\lambda_2)-\nu_k & -P_3(0) \\ 0 & \lambda_1 & \lambda_2 & -P_4(0) \end{vmatrix}. \quad (35)$$

For example it is supposed that $P_1(0) = 1, P_2(0) = 0, P_3(0) = 0, P_4(0) = 0$.

CONCLUSION

The fourth order Kolmogorov equation for state probabilities of a technical system consisting of two subsystems is represented in a symmetrical matrix form. The form is reflected in ordered symmetrical record of characteristic determinant.

The characteristic determinant is expressed in structurally simple formulae for coefficient of dependence-bonded characteristic equation. The dependence demonstrates the nature of symmetry under consideration. Functional dependence of characteristic equation coefficients motivates qualitatively harmonic distribution of roots within complex plane in the form of central symmetry. Specified property of the symmetry makes it possible to apply visible formula to calculate roots of characteristic equation of fourth order.

Analytical solution for Kolmogorov equation is represented in the form of structurally simple and ordered determinants to analyze interactive state probability of technical system.

References

[1]. Lazaryan, V.A., *Dinamyka transportnyh sredstv* [Vehicle dynamics], Kyiv, Naukova dumka, 1985, 528 p. (in Russian).

[2]. Blokhin, E.P., Manashkin, L.A., *Dinamyka poezda* [The dynamics of the train], Moscow, Transport, 1982, 222 p. (in Russian).

[3]. Lysenko, L.N., *Navedenie i navigatsion ballisticheskih raket* [Navigation and guidance ballistic rockets], Moscow, Bauman university publ., 2007, 762 p. (in Russian).

[4]. Igdalov, I.M., et al., *Dinamicheskoe proekrtirovanie raket. Zadachi dinamiki raket I ih kosmicheskih stupeney* [Dynamic design of missiles. The tasks of the dynamics rockets and their cosmic stages], Dnipro, Dnipro national university publ., 2010, 264 p. (in Russian).

[5]. Korn, G., Korn, T., *Spravochnik po matematike dlya nauchnyh rabotnikov i inzhenerov* [Mathematical Handbook for Scientists and Engineers], Moscow, Nauka Publ., 1984, 832 p. (in Russian).

[6]. Smirnov, V.I., *A Course of Higher Mathematics*, Volume IV, New York, Pergamon Press, 1964, 814 p.

[7]. Ovchinnikov, P.P., *Vishha matematika* [Higher mathematics], Part 2, Kyiv, Technika publ., 2002, 512 p. [in Ukrainian].

[8]. Pavlovskij, M.A., *Teoretichna mexanika* [Theoretical mechanics], Kyiv, Technika Publ., 2002, 512 p. [in Ukrainian].

[9]. Ventcel', E.S., Ovcharov, L.A., *Teoriya sluchajnyx protsessov i eyo inzhenernye prilozheniya* [Theory of random processes and its engineering application], Moscow, Nauka Publ., 1991, 384 p. [in Russian].

[10]. Kravets, V.V., et al., *Analytical method of the simplest vehicle component dynamic design in terms of external effect, Scientific messenger of the First Ukrainian Marine Institute*, Issue I, 2014, Sevastopol.: p.79-82. [in Russian].

[11]. Kravets, V., et al., *Analytical modeling of transient process in terms of one-dimensional problem of dynamics with kinematic action, Mechanics, Materials Science & Engineering Journal*, Magnolithe, 2016, 10.13140/RG.2.1.4017.0005. hal-01302014.

[12]. Kravets, V., Chibushov Y., *Method of finding the analytical solution of the algebraic particular aspect equation*, Rzeszow (Poland), *Folia Scient. Univ. Tech. Resoviensis Math.,* No.16, 1994.: p.104-117.

[13]. Kravets, V.V., et al., *Analytical solution of Kolmogorov equations for four-condition homogenous, symmetric and ergodic system,* January 2014, *Open Journal of Applied Sciences,* **04**(10).: p. 497-500, DOI: 10.4236/ojapps.2014.410048.