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# Conversion Among Arma Models and State-Space Representation

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**ABSTRACT**

**We present the ARMA models (or Non-Markovian) and state-space (or Markovian) representation relationship. Then we break the problem into three different cases to discuss how one form could be converted to another form. In case A, we assume that we know the state-space representation then we convert it into the ARMA model. In case B, we reverse the situation, given the ARMA model we convert into state-space representation. In Case C, we combine the first two cases, conversion the two forms in either directions.**

**KEY WORDS:** ARMA Model, Conversion to another form, Markovian Representation, replace t by t+1, replace t+3 by t, State-Space representation, Time Index.

## INTRODUCTION

In discussing linear systems it is often more convenient to use the state-space (or Markovian) representation of the relationship between input and output rather than the explicit form. The state-space form gives a very compact description which is valid provided the relationship between the input and output can be expressed in terms of a finite order linear difference equation. The basic idea rests on the well-known result that any finite order linear difference equation can be expressed as a vector first order equation. For this reason, state-space representation had a profound impact on time series analysis and many related areas. Davis and Vinter(1985) applied these techniques in control of linear systems. Later, Hannan and Deistler (1988) used it in linear system of statistical theory. This is a rich class for time series models and going well beyond the linear ARMA models. In econometrics the structural time series models developed by Harvey (1990) formulated like the classical decomposition model directly in terms of components of interest such as trend, noise. The objective of current paper is not creating a new theory for time series. We wish to build a bridge between ARMA model and state-space representation.

## RELATIONSHIP BETWEEN STATE-SPACE AND ARMA MODEL

Let $x_t$ be a univariate ARMA(p,q) time series with p>q

$$x_t = \phi_1 x_{t-1} + ..... + \phi_p x_{t-p} + a_t - \theta_1 a_{t-1} - .... - \theta_q a_{t-q}$$

We define $x_{t+i|t} = E(x_{t+i} \mid x_t, x_{t-1}....)$ then

$$x_{t+i|t} = x_t \quad \text{for} \quad i \le 0$$

$$a_{t+i|t} = E(a_{t+i} \mid x_t, x_{t-1}.....) = a_{t+i} \qquad i \le 0$$

$$a_{t+i|t} = 0 \qquad \text{for i>0}$$

The state-space formulation is

$$\underline{z}_{t+1} = F\underline{z}_t + Ga_{t+1}$$

where $\underline{z}_t$ is a length p column vector and F is a pxp matrix.

This representation is based on writing

$$\underline{z}_{t+1} = \begin{bmatrix} x_{t+1} \\ x_{t+2|t+1} \\ x_{t+3|t+1} \\ . \\ . \\ x_{t+p|t+1} \end{bmatrix} = \underset{F}{\begin{bmatrix} & & \\ & & \\ & & \\ & & \\ & & \\ & & \end{bmatrix}} \underset{G}{\begin{bmatrix} x_t \\ x_{t+1|t} \\ x_{t+2|t} \\ . \\ . \\ x_{t+p-1|t} \end{bmatrix}} + \begin{bmatrix} \\ \\ \\ \\ \\ \end{bmatrix} a_t$$

The matrices F and G are identified by the following way.

The infinite MA representation for $x_{t+i}$ is

$$x_{t+i} = \sum_{k=0}^{\infty} \Psi_k x_{t+i-k}$$

$$= a_{t+i} + \Psi_1 a_{t+i-1} + .....\Psi_{i-1} a_{t+1} + \sum_{k=i}^{\infty} \Psi_k a_{t+i-k}$$

$$x_{t+i|t} = \sum_{k=i}^{\infty} \Psi_k a_{t+i-k}$$

$$x_{t+i|t+1} = \Psi_{i-1} a_{t+1} + \sum_{k=i}^{\infty} \Psi_k a_{t+i-k}$$

$$= \Psi_{i-1} a_{t+1} + x_{t+i|t} \qquad (2.1)$$

$$x_{t+1} = a_{t+1} + x_{t+1|t} \quad \text{for i=1} \qquad (2.2)$$

Finally, we have the following results.

$$x_{t+p} = \phi_1 x_{t+p-1} + ..... + \phi_p x_t + a_{t+p} - \theta_1 a_{t+p-1} - .... - \theta_q a_{t+p-q}$$

And expectation given $x_t, x_{t-1}$ ...is

$$x_{t+p|t} = \phi_1 x_{t+p-1|t} + ..... + \phi_p x_{t|t} + 0 \qquad \text{since p-q>0}$$

$$x_{t+p|t+1} = \psi_{p-1} a_{t+1} + x_{t+p|t} \qquad (2.3)$$

Summarize (2.1), (2.2), and (2.3), we can write the state-space formulation as below:

$$\begin{bmatrix} x_{t+1} \\ x_{t+2|t+1} \\ x_{t+3|t+1} \\ . \\ . \\ x_{t+p|t+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & . & . & 0 \\ 0 & 0 & 1 & . & . & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & . & . & . & . & 1 \\ \Phi_p & \Phi_{p-1} & & & & \Phi_1 \end{bmatrix} \begin{bmatrix} x_t \\ x_{t+1|t} \\ x_{t+2|t} \\ . \\ . \\ x_{t+p|-1|t} \end{bmatrix} + \begin{bmatrix} 1 \\ \Psi_1 \\ . \\ . \\ . \\ \Psi_{p-1} \end{bmatrix} a_{t+1}$$

The bottom row of the F matrix gives AR coefficients and the G vector gives the first p-1 $\Psi$ coefficients, which can be used to calculates the $\Theta$ coefficients

$$\Phi(B)x_t = \Theta(B)a_t$$

$$x_t = \frac{\Theta(B)}{\Phi(B)}a_t \quad \text{imply} \quad x_t = \Psi(B)a_t$$

$$where \quad \Psi(B) = \frac{\Theta(B)}{\Phi(B)} \text{ so } \Psi(\mathrm{B})\Phi(\mathrm{B}) = \Theta(\mathrm{B})$$

## CASE STUDY

**Case A**. Given the following State-Space representation for a bivariate time series, write it in ARMA Form.

$$\begin{pmatrix} X_{t+1|t+1} \\ X_{t+2|t+1} \\ Y_{t+1|t+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ f_{21} & f_{22} & f_{23} \\ 0 & 0 & f_{33} \end{pmatrix} \begin{pmatrix} X_{t|t} \\ X_{t+1|t} \\ Y_{t|t} \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ g_1 & g_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_{1,t+1} \\ a_{2,t+1} \end{pmatrix}$$

$$X_{t+1|t+1} = X_{t+1|t} + a_{1,t+1} \qquad (3a.1)$$

$$X_{t+2|t+1} = f_{21}X_{t|t} + f_{22}X_{t+1|t} + f_{23}Y_{t|t} + g_1a_{1,t+1} + g_2a_{2,t+1} \qquad (3a.2)$$

$$\mathrm{y}_{\mathrm{t+1|t+1}} = f_{33}y_{t|t} + a_{2,t+1} \qquad (3a.3)$$

From (3a.1), rewrite as

$$X_{t+1|t} = X_{t+1} - a_{1,t+1} \qquad (3a.4)$$

Substitute (3a.4) into (3a.2)

$$X_{t+2} - a_{1,t+2} = f_{21}X_t + f_{22}(X_{t+1} - a_{1,t+1}) + f_{23}Y_t + g_1a_{1,t+1} + g_2a_{2,t+1} \qquad (3a.5)$$

Rewrite (3a.5) in the form

$$(1 - f_{22}B - f_{21}B^2)X_{t+2} - f_{23}B^2Y_{t+2} = (1 + (g_1 - f_{22})B)a_{1,t+2} + g_2Ba_{2,t+2} \qquad (3a.6)$$

From (3a.3), we replace t by t+1 and have

$$y_{t+2} = f_{33} y_{t+1} + a_{2,t+2}$$

Rewrite above equation in the form

$$(1-f_{33}B)y_{t+2} = a_{2,t+2} \qquad (3a.7)$$

Combine (3a.6) and (3a.7), we write in bivariate time series form,

$$\begin{pmatrix} 1-f_{22}B-f_{21}B^2 & -f_{23}B^2 \\ 0 & 1-f_{33}B \end{pmatrix}\begin{pmatrix} x_{t+2} \\ y_{t+2} \end{pmatrix} = \begin{pmatrix} 1+(g_1-f_{22})B & g_2B \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a_{1,t+2} \\ a_{2,t+2} \end{pmatrix}$$

$$\left(I-\Phi_1 B-\Phi_2 B^2\right)\begin{pmatrix} x_{t+2} \\ y_{t+2} \end{pmatrix} = \left(I-\Theta_1 B\right)\begin{pmatrix} a_{1,t+2} \\ a_{2,t+2} \end{pmatrix}$$

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \Phi_1 = \begin{pmatrix} f_{22} & 0 \\ 0 & f_{33} \end{pmatrix}, \quad \Phi_2 = \begin{pmatrix} f_{21} & f_{23} \\ 0 & 0 \end{pmatrix}$$

$$\Theta_1 = \begin{pmatrix} -(g_1-f_{22}) & -g_2 \\ 0 & 0 \end{pmatrix}$$

where

**Case B.** Write the following bivariate ARMA (1,0,1) model in the State-Space formulation.

$$(I-\begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix}B)\begin{pmatrix} X_t \\ Y_t \end{pmatrix} = \left(I-\begin{pmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{pmatrix}B\right)\begin{pmatrix} a_{1t} \\ a_{2t} \end{pmatrix}$$

$$\begin{pmatrix} 1-\phi_{11}B & -\phi_{12}B \\ -\phi_{21}B & 1-\phi_{22}B \end{pmatrix}\begin{pmatrix} X_t \\ Y_t \end{pmatrix} = \left(\begin{pmatrix} 1-\theta_{11}B & -\theta_{12}B \\ -\theta_{21}B & 1-\theta_{22}B \end{pmatrix}\right)\begin{pmatrix} a_{1t} \\ a_{2t} \end{pmatrix}$$

$$X_t = \phi_{11}X_{t-1} + \phi_{12}Y_{t-1} + a_{1,t} - \theta_{11}a_{1,t-1} - \theta_{12}a_{2,t-1} \qquad (3b.1)$$

$$Y_t = \phi_{22}Y_{t-1} + \phi_{21}X_{t-1} + a_{2,t} - \theta_{22}a_{2,t-1} - \theta_{21}a_{1,t-1} \qquad (3b.2)$$

From (3b.1), we replace t by t+1 and consider the conditional expectation up to time t

$$X_{t+1|t} = \phi_{11}X_t + \phi_{12}Y_t - \theta_{11}a_{1,t} - \theta_{12}a_{2,t} \qquad (3b.3)$$

Same reason for equation (3b.2)

$$Y_{t+1|t} = \phi_{22}Y_t + \phi_{21}X_t - \theta_{22}a_{2,t} - \theta_{21}a_{1,t} \qquad (3b.4)$$

If we replace t by t+2 then consider the conditional expectation up to t, we have

$$X_{t+2|t} = \phi_{11}X_{t+1|t} + \phi_{12}Y_{t+1|t} \qquad (3b.5)$$

$$Y_{t+2|t} = \phi_{22}Y_{t+1|t} + \phi_{21}X_{t+1|t} \qquad (3b.6)$$

Both of the (3b.5) and (3b.6) are linear combination of the (3b.3) and (3b.4). Hence it should not show in State-space formula. From (3b.1) and (3b.2), we have the Following results.

$$X_{t+1} = X_{t+1|t} + a_{1,t+1} \qquad (3b.7)$$

$$Y_{t+1} = Y_{t+1|t} + a_{2,t+1} \qquad (3b.8)$$

Substitute (3b.7) and (3b.8) into (3b.3) and (3b.4), replace t by t+1 then we can derive

$$\begin{aligned} X_{t+2|t+1} &= \phi_{11}(X_{t+1|t} + a_{1,t+1}) + \phi_{12}(Y_{t+1|t} + a_{2,t+1}) - \theta_{11}a_{1,t+1} - \theta_{12}a_{2,t+1} \\ &= \phi_{11}X_{t+1|t} + \phi_{12}Y_{t+1|t} + (\phi_{11} - \theta_{11})a_{1,t+1} + (\phi_{12} - \theta_{12})a_{2,t+1} \qquad (3b.9) \end{aligned}$$

$$\begin{aligned} Y_{t+2|t+1} &= \phi_{22}(Y_{t+1|t} + a_{2,t+1}) + \phi_{21}(X_{t+1|t} + a_{1,t+1}) - \theta_{22}a_{2,t+1} - \theta_{21}a_{1,t+1} \\ &= \phi_{22}Y_{t+1|t} + \phi_{21}X_{t+1|t} + (\phi_{22} - \theta_{22})a_{2,t+1} + (\phi_{21} - \theta_{21})a_{1,t+1} \qquad (3b.10) \end{aligned}$$

Combine (3b.7), (3b.8), (3b.9) and (3b.10), we could formulate State-Space model as follow:

$$\begin{pmatrix} X_{t+1|t+1} \\ Y_{t+1|t+1} \\ X_{t+2|t+1} \\ Y_{t+2|t+1} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \phi_{11} & \phi_{12} \\ 0 & 0 & \phi_{21} & \phi_{22} \end{pmatrix} \begin{pmatrix} X_t \\ Y_t \\ X_{t+1|t} \\ Y_{t+1|t} \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \phi_{11} - \theta_{11} & \phi_{12} - \theta_{12} \\ \phi_{21} - \theta_{21} & \phi_{22} - \theta_{22} \end{pmatrix} \begin{pmatrix} a_{1,t+1} \\ a_{2,t+1} \end{pmatrix}$$

**Case C.** Write the ARMA (2,0,2) model in the state-space formulation then convert it back to ARMA (2,0,2) model.

$$(1 - \phi_1 B - \phi_2 B^2)Z_t = (1 - \theta_1 B - \theta_2 B^2)a_t$$

$$Z_t = (1 - \phi_1 B - \phi_2 B^2)^{-1}(1 - \theta_1 B - \theta_2 B^2)a_t = \sum_{k=0}^{\infty} \psi_k a_{t-k} \qquad (3c.1)$$

state-space representation

$$\begin{pmatrix} Z_{t+1|t+1} \\ Z_{t+2|t+1} \\ Z_{t+3|t+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & \phi_2 & \phi_1 \end{pmatrix} \begin{pmatrix} Z_t \\ Z_{t+1|t} \\ Z_{t+2|t} \end{pmatrix} + \begin{pmatrix} 1 \\ \psi_1 \\ \psi_2 \end{pmatrix} a_{t+1} \qquad (3c.2)$$

Spell out (3c.2), we have

$$Z_{t+1|t+1} = Z_{t+1|t} + a_{t+1} \qquad (3c.3)$$

$$Z_{t+2|t+1} = Z_{t+2|t} + \psi_1 a_{t+1} \qquad (3c.4)$$

$$Z_{t+3|t+1} = \phi_2 Z_{t+1|t} + \phi_1 Z_{t+2|t} + \psi_2 a_{t+1} \qquad (3c.5)$$

From (3c.3) and (3c.4), we replace t by t+1 and we derive

$$Z_{t+2|t+1} = Z_{t+2|t+2} - a_{t+2} \qquad (3c.6)$$

$$Z_{t+3|t+1} = Z_{t+3|t+2} - \psi_1 a_{t+2} \qquad (3c.7)$$

substitute (3c.6) to right side of (3c.7)

$$Z_{t+3|t+1} = Z_{t+3|t+3} - a_{t+3} - \psi_1 a_{t+2} \qquad (3c.8)$$

Substitute (3c.8), (3c.3) and (3c.4) into (3c.5), we derive

$$Z_{t+3} = a_{t+3} + \psi_1 a_{t+2} + \phi_2 (Z_{t+1} - a_{t+1}) + \phi_1 (Z_{t+2} - a_{t+2} - \psi_1 a_{t+1}) + \psi_2 a_{t+1} \qquad (3c.9)$$

Lastly, we replace t+3 by t in (3c.9) and rearrange order

$$\begin{aligned} Z_t &= a_t + \psi_1 a_{t-1} + \phi_2 (Z_{t-2} - a_{t-2}) + \phi_1 (Z_{t-1} - a_{t-1} - \psi_1 a_{t-2}) + \psi_2 a_{t-2} \\ &= \phi_2 Z_{t-2} + \phi_1 Z_{t-1} + a_t + (\psi_1 - \phi_1) a_{t-1} + (\psi_2 - \phi_1 \psi_1 - \phi_2) a_{t-2} \end{aligned} \qquad (3c.10)$$

At last, we can write (3c.10) in the ARMA (2,0,2) model

$$(1 - \phi_1 B - \phi_2 B^2) Z_t = (1 - \theta_1 B - \theta_2 B^2) a_t$$

$$where \quad -\theta_1 = (\psi_1 - \phi_1) \quad -\theta_2 = (\psi_2 - \phi_1 \psi_1 - \phi_2)$$

In above equation we had applied the relationship among $\theta_i, \psi_i$ and $\phi_i$. It needs some algebraic process to derive these results. We leave it in the appendix.

## CONCLUDING REMARKS

In control theory of linear systems or in econometrics, these two forms played the central key roles. In this paper, we present a bridge to link the ARMA models and state space representation in such a way even a researcher does not familiar with time series theories can switch one form to the other. So that he can plug in the needed form into SAS package to complete the computing. We had chosen our notation consistent with the SAS package to minimize the possible confusion. It seems much more straightforward to convert from ARMA model to state-space representation. However, the process needs more careful on two tricky factors. We must be certain that all the variables are independent to each other. Secondly, the time index must be carefully treated.

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## APPENDIX

$$Z_t = (1-\phi_1 B-\phi_2 B^2)^{-1}(1-\theta_1 B-\theta_2 B^2)a_t = \sum_{k=0}^{\infty}\psi_k a_{t-k}$$

$$(1-\theta_1 B-\theta_2 B^2)a_t = (1-\phi_1 B-\phi_2 B^2)(1+\psi_1 B+\psi_2 B^2+\psi_3 B^3+.....)a_t$$

$$= (1+(\psi_1-\phi_1)B+(\psi_2-\psi_1\phi_1-\phi_2)B^2+...)a_t$$

$$(\psi_1-\phi_1) = -\theta_1 \quad , \quad (\psi_2-\phi_1\psi_1-\phi_2) = -\theta_2$$