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New Formula for Solution of One Class of Linear Differential Equations of the Second Order with the Variable Coefficients

[1]Avyt Asanov, [2]Kanykei Asanova
[1]Kyrgyz-Turkish Manas University, Bishkek, Kyrgyzstan;
[2]Institute of Mathematics, National Academy of Sciences of Kyrgyz Republic, Bishkek, Kyrgyzstan;
avyt.asanov@manas.edu.kg; avyt.asanov@mail.ru; kanya.asanova@gmail.com

ABSTRACT

Exact solutions for linear and nonlinear differential equations play an important role in theoretical and practical research. In particular many works have been devoted to finding a formula for solving second order linear differential equations with variable coefficients. In this paper we obtained the formula for the common solution of the linear differential equation of the second order with the variable coefficients in the more common case. We also obtained the new formula for the solution of the Cauchy problem for the linear differential equation of the second order with the variable coefficients.Examples illustrating the application of the obtained formula for solving second-order linear differential equations are given.

Key words: The linear differential equation, the second order, the variable coefficients,the new formula for the common solution, Cauchy problem, examples.

1 Introduction

We consider the equation

$$L[y] \equiv y'' + p(t)y' + q(t)y = f(t), t \in I, \quad (1)$$

where $I = (t_1, t_2)$ or $I = [t_1, t_2]$ or $I = [t_1, t_2)$ or $(t_1, t_2], t_1 < t_2$, $p(t), q(t)$ and $f(t)$ are known continuous functions on I.

Many works [1-6] are dedicated to the determination of the common solutions of the linear and nonlinear ordinary differential equations. Exact solutions of differential equations play an important role in the proper understanding of qualitative features of many phenomena and processes in various areas of natural science.These solutions can be used to verify the consistencies end estimate errors of various numerical, asymptotic, and approximate analytical methods.In particular, formulas for solving various linear and nonlinear differential equations are obtained in [1]. In [2-3] are dedicated to the determination of the solutions of the first order and the second order ordinary differential equations.But in common case any formulas for the decision of the linear differential equations of the second order haven't obtained. It is well known that if $p(t) = p_0 = const$, $q(t) = q_0 = const$, then depending on the sign of discriminant $D = p_0^2 - 4q_0$ the common solution of the equation (1) will be written by three formulas. In [5] and [6] the general formula of the solution for linear ordinary differential equations of the second

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order with variable coefficients generalizing respectively $D = p_0^2 - 4q_0 > 0$ and $D = p_0^2 - 4q_0 < 0$ are received. In this theme the equation (1) is investigated in the more common cases. Depending on the correlation between *p(t)* and *q(t)* formulas for the determination of the common solution of this equation were obtained. It is shown in the examples that the obtained formulas generalize many known formulas obtained in [1], [5] and [6].

2 The Linear Differential Equations of the Second Order

Consider the linear differential equations of the second order (1).

Theorem 1. Let

$$q(t) = a(t)\psi(t)[p(t) - a(t)\psi(t)] + a(t)\psi'(t) + \psi^2(t)l^2(t) + a'(t)\psi(t), t \in I, \tag{2}$$

$$l(t) = r\exp\left\{\int [2\psi(t)a(t) - p(t) - (\psi(t))^{-1}\psi'(t)]dt\right\}, \tag{3}$$

where $a(t),\ a'(t),\ p(t), \psi(t), \psi'(t),\ f(t) \in C(I), r \in R, a'(t)$ and $\psi'(t)$ are respectively the derivatives of the functions $a(t)$ and $\psi(t)$. Then:

1) If r = 0, then the common solution of the equation (1) will be written in the next form

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + y_3(t), t \in I \tag{4}$$

where c_1 and c_2 are arbitrary constants,

$$y_1(t) = \exp\left[-\int_{t_0}^{t} a(s)\psi(s)ds\right], \quad t_0 \in I, \tag{5}$$

$$y_2(t) = y_1(t)\int_{t_0}^{t} (y_1^2(s))^{-1} e^{-\int_{t_0}^{s} p(\tau)d\tau} ds, \tag{6}$$

$$y_3(t) = \int_{t_0}^{t} y_1(t)y_1(s)f(s)\left[\int_{s}^{t} (y_1^2(\tau))^{-1} e^{-\int_{\tau}^{s} p(\tau)d\tau} d\tau\right] ds, \tag{7}$$

$$\Delta_0(t) = \det\begin{pmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{pmatrix} = e^{-\int_{t_o}^{t} p(s)ds}, \quad t_0 \in I.$$

2) If $r \neq 0$ and $\psi(t) \neq 0$ for all $t \in I$, then the common solution of the equation (1) will be written in the next form

$$y(t) = c_1 z_1(t) + c_2 z_2(t) + z_3(t), t \in I, \tag{8}$$

where c_1 and c_2 are arbitrary constants,

$$z_1(t) = \exp\left[-\int_{t_0}^{t} a(s)\psi(s)ds\right]\cos\left[\int_{t_0}^{t} l(s)\psi(s)ds\right], \quad t \in I, \tag{9}$$

$$z_2(t) = \exp\left[-\int_{t_0}^{t} a(s)\psi(s)ds\right]\sin\left[\int_{t_0}^{t} l(s)\psi(s)ds\right], \tag{10}$$

$$z_3(t) = \int_{t_0}^{t} \frac{f(s)}{l(s)\psi(s)} e^{-\int_{s}^{t} a(\tau)\psi(\tau)d\tau} \sin\left[\int_{s}^{t} l(\tau)\psi(\tau)d\tau\right] ds, \tag{11}$$

$$\Delta(t) = \det\begin{pmatrix} z_1(t) & z_2(t) \\ z_1'(t) & z_2'(t) \end{pmatrix} = l(t)\psi(t)e^{-2\int_{t_0}^{t} a(\tau)\psi(\tau)d\tau}, \quad t \in I.$$

Proof. 1) We show that

$$L[y_1] = 0,\ L[y_2] = 0,\ L[y_3] = f(t),\ t \in I.$$

At first we prove $L[y_1] = 0$. In fact if we differentiate (5) we will obtain

$$y_1'(t) = -a(t)\psi(t)y_1(t), \tag{12}$$

$$y_1''(t) = y_1(t)[a^2(t)\psi^2(t) - a'(t)\psi(t) - a(t)\psi'(t)]. \tag{13}$$

Then taking into account (12), (13) and (2) we have

$$L[y_1] = y_1(t)[a^2(t)\psi^2(t) - a'(t)\psi(t) - a(t)\psi'(t) - p(t)a(t)\psi(t) + q(t)] = 0, t \in I.$$

We show that $L[y_2] = 0$. If we differentiate (6) we will have

$$y_2'(t) = -a(t)\psi(t)y_2(t) + e^{-\int_{t_0}^{t}[p(s)-a(s)\psi(s)]ds}, \tag{14}$$

$$y_2''(t) = y_2(t)[a^2(t)\psi^2(t) - a'(t)\psi(t) - a(t)\psi'(t)] - e^{-\int_{t_0}^{t}[p(s)-a(s)\psi(s)]ds} p(t). \tag{15}$$

Then taking into account (14), (15) and (2) we have

$$L[y_2] = y_2(t)[a^2(t)\psi^2(t) - a'(t)\psi(t) - a(t)\psi'(t) - p(t)a(t)\psi(t) + q(t)] +$$

$$+e^{-\int_{t_0}^{t}[p(s)-a(s)\psi(s)]ds}[-p(t) + p(t)] = 0, t \in I.$$

We are going to prove $L[y_3] = f(t), t \in I$. Differentiating (7) we have

$$y_3'(t) = -a(t)\varphi(t)y_3(t) + \int_{t_0}^{t} e^{-\int_{s}^{t} p(\tau)d\tau} (y_1(t))^{-1} y_1(s) f(s)ds, \tag{16}$$

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$$y_3^{''}(t) = y_3(t)[a^2(t)\psi^2(t) - a'(t)\psi(t) - a(t)\psi'(t)] +$$
$$+[-a(t)\psi(t) - p(t)]\int_{t_0}^{t} e^{-\int_s^t p(\tau)d\tau} (y_1(t))^{-1} y_1(s) f(s)ds + \quad (17)$$
$$+ f(t) + a(t)\psi(t)\int_{t_0}^{t} e^{-\int_s^t p(\tau)d\tau} (y_1(t))^{-1} y_1(s) f(s)ds.$$

Taking into account (16), (17) and (2) we have

$$L[y_3] = y_3(t)[a^2(t)\psi^2(t) - a'(t)\psi(t) - a(t)\psi'(t) - a(t)\psi(t)p(t) + q(t)] + f(t) +$$
$$+[-p(t) + p(t)]\int_{t_0}^{t} e^{-\int_s^t p(\tau)d\tau} (y_1(t))^{-1} y_1(s) f(s)ds = f(t), t \in I.$$

From (5), (6), (12) and (14) we have

$$\Delta_0(t) = e^{-\int_{t_o}^{t} p(s)ds}, \quad t \in I.$$

2) We show that

$$L[z_1] = 0,\ L[z_2] = 0,\ L[z_3] = f(t), t \in I.$$

At first we prove $L[z_1] = 0$. In fact if we differentiate (9)

$$z_1'(t) = -a(t)\psi(t)z_1(t) - l(t)\psi(t)z_2(t), \quad (18)$$

$$z_1^{''}(t) = z_1(t)[a^2(t)\psi^2(t) - a'(t)\psi(t) - a(t)\psi'(t) - l^2(t)\psi^2(t)] +$$
$$+z_2(t)[2a(t)l(t)\psi^2(t) - l'(t)\psi(t) - l(t)\psi'(t)], \quad (19)$$

$$l'(t) = [2\psi(t)a(t) - p(t) - (\psi(t))^{-1}\psi'(t)]l(t). \quad (20)$$

Then taking into account (18), (19), (20) and (2) we have

$$L[z_1] = z_1(t)[a^2(t)\psi^2(t) - a'(t)\psi(t) - a(t)\psi'(t) - l^2(t)\psi^2(t) - p(t)a(t)\psi(t) + q(t)] +$$
$$+z_2(t)\{2a(t)l(t)\psi^2(t) - [2\psi(t)a(t) - p(t) - (\psi(t))^{-1}\psi'(t)]l(t)\psi(t) -$$
$$-l(t)\psi'(t) - p(t)l(t)\psi(t)\} = 0, t \in I.$$

We show $L[z_2] = 0$. If we differentiate (10) we will have

$$z_2'(t) = -a(t)\psi(t)z_2(t) + l(t)\psi(t)z_1(t), \quad (21)$$

$$z_2^{''}(t) = z_2(t)[a^2(t)\psi^2(t) - l^2(t)\psi^2(t) - a'(t)\psi(t) - a(t)\psi'(t)] +$$
$$z_1(t)[-2a(t)l(t)\psi^2(t) + l'(t)\psi(t) + l(t)\psi'(t)]. \quad (22)$$

Then taking info account (21), (22), (20) and (2) we have

$$L[z_2] = z_2(t)[a^2(t)\psi^2(t) - l^2(t)\psi^2(t) - a'(t)\psi(t) - a(t)\psi'(t) - p(t)a(t)\psi(t) + q(t)] +$$
$$+z_1(t)\{-2a(t)l(t)\psi^2(t) + [2\psi(t)a(t) - p(t) - (\psi(t))^{-1}\psi'(t)]l(t)\psi(t) +$$
$$+l(t)\psi'(t) + p(t)l(t)\psi(t)\} = 0, t \in I.$$

We are going to prove $L[z_3] = f(t), t \in I$. Differentiating (11) we have

$$z_3'(t) = -a(t)\psi(t)z_3(t) + l(t)\psi(t)\int_{t_0}^{t} \frac{f(s)}{l(s)\psi(s)} e^{-\int_s^t a(\tau)\psi(\tau)d\tau} \cos\left[\int_s^t l(\tau)\psi(\tau)d\tau\right] ds, \quad (23)$$

$$z_3''(t) = z_3(t)[a^2(t)\psi^2(t) - a'(t)\psi(t) - a(t)\psi'(t) - l^2(t)\psi^2(t)] +$$
$$+\left\{\int_{t_0}^{t} \frac{f(s)}{l(s)\psi(s)} e^{-\int_s^t a(\tau)\psi(\tau)d\tau} \cos\left[\int_s^t l(\tau)\psi(\tau)d\tau\right] ds\right\}[-2a(t)l(t)\psi^2(t) + \quad (24)$$
$$+l'(t)\psi(t) + l(t)\psi'(t)] + f(t).$$

Then taking into account (23), (24), (20) and (2) we have

$$L[z_3(t)] = z_3(t)[a^2(t)\psi^2(t) - a'(t)\psi(t) - a(t)\psi'(t) - l^2(t)\psi^2(t) - p(t)a(t)\psi(t) + q(t)] +$$
$$+\{-2a(t)l(t)\psi^2(t) + [2\psi(t)a(t) - p(t) - (\psi(t))^{-1}\psi'(t)]l(t)\psi(t) + l(t)\psi'(t) +$$
$$+p(t)l(t)\psi(t)\}\int_{t_0}^{t} \frac{f(s)}{l(s)\psi(s)} e^{-\int_s^t a(\tau)\psi(\tau)d\tau} \cos\left[\int_s^t l(\tau)\psi(\tau)d\tau\right] ds + f(t) = f(t), t \in I.$$

Theorem 1 has been proved.

Corollary. Let $t_0 \in I$ and suppose that the conditions of theorem 1 hold. Then:

1) if $r = 0$, then solution of the equation (1) with the initial condition

$$y(t_0) = m,\ y'(t_0) = n,\ m, n \in R \quad (25)$$

will be written in the next form

$$y(t) = my_1(t) + [n + ma(t_0)\psi(t_0)]y_2(t) + y_3(t),\ t \in I, \quad (26)$$

where the functions $y_1(t), y_2(t)$ and $y_3(t)$ are defined by the formulas (5), (6) and (7).

2) if $r \neq 0$ and $\psi(t) \neq 0$ for all $t \in I$, then solution of the equation (1) with the initial equation (25) will be written in the next form

$$y(t) = mz_1(t) + \frac{n + ma(t_0)\psi(t_0)}{l(t_0)\psi(t_0)} z_2(t) + z_3(t),\ t \in I, \quad (27)$$

where the function $z_1(t), z_2(t)$ and $z_3(t)$ are defined by the formulas (9), (10) and (11).

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3 Examples

Example 1. We consider the equation (1) for $p(t) = p \in R, q(t) = q \in R, p^2 - 4q > 0$. In this case for

$$a(t) = a = \frac{p + \sqrt{p^2 - 4q}}{2}, \psi(t) = 1, r = 0,$$

all conditions of theorem 1 hold.

Example 2. We consider the equation (1) for $p(t) = p \in R, q(t) = q \in R, p^2 - 4q = 0$ In this case for $a(t) = \frac{p}{2}, \psi(t) = 1, r = 0, q(t) = \frac{p^2}{4}$ all conditions of theorem 1 hold.

Example 3. We consider the equation (1) for $p(t) = p \in R, q(t) = q \in R, p^2 - 4q < 0$. In this case for $\psi(t) = 1, a(t) = \frac{p}{2}, l(t) = r, r^2 = q - \frac{p^2}{4}$ all conditions of theorem 1 hold.

Example 4. We consider the equation 2.1.2.32 in [1], i.e. we consider the equation (1) for $p(t) = at + b, q(t) = c(-ct^{2n} + at^{n+1} + bt^n + nt^{n-1}), a, b, c \in R.$ In this case for $a(t)\psi(t) = ct^n, r = 0$ all conditions of theorem 1 hold.

Example 5. We consider the equation 2.1.2.59 in [1], i.e. we consider the equation (1) for $p(t) = at^n + bt^m, q(t) = abt^{m+n} + b(m+1)t^{m-1} - at^{n-1}, a, b \in R.$ In this case for $a(t)\psi(t) = bt^m - \frac{1}{t}, r = 0$ all conditions of theorem 1 hold.

Example 6. We consider the equation 2.1.2.60 in [1], i.e. we consider the equation (1) for $p(t) = at^n + bt^m + c, q(t) = abt^{m+n} + bct^m + ant^{n-1}, a, b, c \in R.$ In this case for $a(t)\psi(t) = at^n + c, r = 0$ all conditions of theorem 1 hold.

Example 7. We consider the equation 2.1.2.155 in [1], i.e. we consider the equation (1) for $p(t) = -\frac{3t}{1-t^2}, q(t) = \frac{n(n+2)}{1-t^2}, n \in N.$ In this case for $a(t)\psi(t) = -\frac{t}{1-t^2}, l(t)\psi(t) = -\frac{n+1}{\sqrt{1-t^2}}, r = n+1, I = (-1,1)$ all conditions of theorem 1 hold.

Example 8. We consider the equation 2.1.8.19 in [1], i.e. we consider the equation (1) for $p(t) = a\sin^n(kt), q(t) = be^{\lambda t}[a\sin^n(kt) - be^{\lambda t} + \lambda], a, b, \lambda \in R.$ In this case for $a(t)\psi(t) = be^{\lambda t}, r = 0$ all conditions of theorem 1 hold.

Example 9. We consider the equation 2.1.9.61 in [1], i.e. we consider the equation (1) for $p(t) = f(t) + g(t), q(t) = f(t)g(t) + f'(t).$ In this case for $a(t)\psi(t) = f(t), r = 0$ all conditions of theorem 1 hold.

Example 10. We consider the equation 2.1.2.81 in [1], i.e. we consider the equation (1) for $p(t) = -f'(t), q(t) = b^2 e^{2f(t)}, b \in R.$ In this case for $r \neq 0, a(t) \equiv 0, l(t)\psi(t) = be^{f(t)}$ all conditions of theorem 1 hold.

Example 11. We consider the equation 2.1.9.82 in [1], i.e. we consider the equation (1) for $p(t) = -f'(t), q(t) = -b^2 e^{2f(t)}, b \in R.$ In this case for $r = 0, a(t)\psi(t) = be^{f(t)}$, all conditions of theorem 1 hold.

Example 12. We consider the equation 2.1.9.84 in [1], i.e. we consider the equation (1) for $p(t) = 0, q(t) = -\frac{f''(t)}{2f(t)} + \left(\frac{f'(t)}{2f(t)}\right)^2 - \frac{b}{4f^2(t)}, b \in R.$ za. Then:

1) If $b > 0$, then for $r = 0, a(t)\psi(t) = \frac{\sqrt{b}}{2f(t)} - \frac{f'(t)}{2f(t)}$ all conditions of theorem 1 hold;

2) If $b < 0$, then for $r \neq 0, a(t)\psi(t) = -\frac{f'(t)}{2f(t)}, l(t)\psi(t) = \frac{1}{2}\sqrt{|b|}\frac{1}{f(t)}$ all conditions of theorem 1 hold;

3) If $b = 0$, then for $r = 0, a(t)\psi(t) = -\frac{f'(t)}{2f(t)}$ all conditions of theorem hold.

Example 13. Let for the equation (1) all conditions of the theorem 1 hold and

$$r = 0, \psi(t) = \varphi'(t), a(t) = a(t) + b(t)e^{\int \alpha(t)\varphi'(t)dt}, t \in I,$$

where $\varphi'(t), \alpha(t), b(t), \varphi''(t), \alpha'(t), b'(t) \in C(I), \varphi'(t) \neq 0$ and $b(t) \neq 0$ for all $t \in I$. Then from (2) we have

$$q(t) = (\varphi'(t))^2 e^{\int \alpha(t)\varphi'(t)dt} \left\{ b'(t)(\varphi'(t))^{-1} - b^2(t)e^{\int \alpha(t)\varphi'(t)dt} + \right.$$
$$\left. + \left[p(t) + \frac{\varphi''(t)}{\varphi'(t)} \right] (\varphi'(t))^{-1} b(t) - \alpha(t)b(t) \right\} +$$
$$+\varphi'(t))^2 \left\{ \alpha'(t)(\varphi'(t))^{-1} - \alpha^2(t) + \alpha(t) \left[p(t) + \frac{\varphi''(t)}{\varphi'(t)} \right] (\varphi'(t))^{-1} \right\}, t \in I,$$

i.e. all conditions of the theorem 1 in the work [5] hold.

Example 14. Let for the equation (1) all conditions of the theorem 1 hold and

$$a(t) = \frac{K(t)}{\psi(t)}, l(t) = \frac{\beta(t)}{\psi(t)}, p(t) = 2K(t) - \frac{\beta'(t)}{\beta(t)}, t \in I,$$

where $K(t), \beta(t), \psi(t), K'(t), \beta'(t), \psi'(t) \in C(I), \psi(t) \neq 0$ and $\beta(t) \neq 0$ for all $t \in I$. Then from (2) we have

$$q(t) = K^2(t) + \beta^2(t) + K'(t) - \frac{\beta'(t)}{\beta(t)} K(t), t \in I,$$

i.e. all conditions of the theorem 1 in the work [6] hold.

3 Conclusion

It is known that linear differential equations of the second order are applied in various applied and theoretical problems.In these circumstances, an important role is played by the formula for the solution of linear differential equations of second order. We hope that the new formula obtained with us will help many researchers in their research**.**

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