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Index Theory and Multiple Solutions for a Fractional Laplacian Equation with an Asymptotically Linear Term

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ABSTRACT

In this paper, we investigate the existence and multiplicity of solutions for the asymptotically linear fractional Laplacian equation with homogeneous Dirichlet boundary conditions. We will construct an index theory for the associated linear fractional Laplacian equation. Using results from critical point theory, we show how the behavior of the nonlinearity near zero and at infinity affects the number of solutions via the index.

Keywords: Fractional Laplacian equation, Homogeneous Dirichlet boundary condition, Index theory, Variational techniques.

INTRODUCTION AND MAIN RESULTS

The goal of this paper is to establish the existence and multiplicity of solutions to the following nonlocal elliptic equation

$$\begin{aligned} (-\Delta)^s u &= f(x,u) \text{ in } \Omega\,, \\ u &= 0 \text{ in } \mathbb{R}^N \backslash \Omega\,, \end{aligned} \tag{1.1}$$

where $s \in (0,1)$ is fixed, $N > 2s$, $\Omega \subset \mathbb{R}^N$ is an open bounded set with Lipschitz boundary. Here, $(-\Delta)^s$ is the fractional Laplace operator, which (up to normalization factors) is defined as

$$-(-\Delta)^s u(x) = \int_{\mathbb{R}^N} \frac{u(x+y)+u(x-y)-2u(x)}{|y|^{N+2s}}\, dy,\ x \in \mathbb{R}^N.$$

The nonlocal equations have been experiencing impressive applications in different subjects, such as the thin obstacle problem, phase transitions, stratified materials, anomalous diffusion, crystal dislocation, soft thin films, semipermeable membranes and flame propagation, conservation laws, ultrarelativistic limits of quantum mechanics, quasigeostrophic flows, multiple scattering, minimal surfaces, materials science, water waves, elliptic problems with measure data, optimization, finance, etc. See [24] and the references therein.

In the literature, many papers are devoted to the study of the non-local fractional Laplacian equation (1.1) with nonlinearities having subcritical or critical growth, see [2] [3], [4], [5], [6][7] [8] [22][23][24],[26] [27] [28] [29] [30] and references therein. In this paper, we assume that the nonlinearity f is continuous in $\overline{\Omega} \times \mathbb{R}$ and verify the following conditions:

(f_0) $f(x,u) = B_0(x)u + f_1(x,u)$, with B_0 a continuous function in $\overline{\Omega}$ and $f_1(x,u) = o(|u|)$ uniformly in x as $|u| \to 0$.

(f_∞) $f(x,u) = B_\infty(x)u + f_2(x,u)$, with B_∞ a continuous function in $\overline{\Omega}$ and $f_2(x,u) = o(|u|)$ uniformly in x as $|u| \to \infty$.

Conditions (f_0) and (f_∞) are referred to in the literature as asymptotically linear for the nonlinearity at the origin and at infinity. A quantitative way to measure the difference between B_0 and B_∞ is given by the index, which was introduced in [9][12][17] [18] [19] [20] [33] [34] by Amann, Zehnder, Ekeland and Long, and developed in [10] [11] [16] [31] for the study of Hamiltonian system in relation with Morse theory and critical point theory. We refer to the books of Ekeland [13] and Long [21] for a more detailed account of the concept. We think that a natural question is whether or not these index theories may be adapted to the fractional Laplacian equation Eq (1.1). The aim of this paper is to consider extending the index theory to fractional Laplacian equation Eq (1.1) (see Section 2.2) and show how the behavior of the nonlinearity f at the origin and at the infinity affects the number of solutions.

In order to state our main result of this paper, we briefly introduce the index theory for the linear fractional Schrödinger equation. For any function $B(x)$ continuous in Ω, the index is a pair of integers, denoted by $(i(B), v(B))$ where $v(B)$ is the dimension of the solution space of $(-\Delta)^s u - B(x)u = 0$ (see Lemma 2.3), and $i(B)$ is the dimension of the eigenvector space associated with all the negative eigenvalues (see Lemma 2.4). Moreover, $i(B)$ is nondecreasing with respect to B.

Our results read as follows.

Theorem 1.1. *Let (f_0) and (f_∞) satisfy $v(B_0) = 0$ and $v(B_\infty) = 0$. If $f(x,u)$ is odd in u, then Eq. (1.1) has at least $|i(B_\infty) - i(B_0)|$ pairs of solutions.*

Remark 1.1. *(1) It is worth pointing that the condition $v(B_\infty) = 0$ means the problem (1.1) is non-resonant at infinity. For the resonant case, we refer to* [14].

(2) Consider the following eigenvalue problem

$$(-\Delta)^s u = \lambda u \text{ in } \Omega, \tag{1.2}$$

$$u = 0 \text{ in } \mathbb{R}^N \backslash \Omega.$$

Let the eigenvalues of this problem be denoted by

$$\lambda_1 < \lambda_2 < \lambda_3 \cdots$$

counting their multiplicity. Let $dim(\lambda_i)$ denote the dimension of the eigenvector space associate with the eigenvalue λ_i. Set $d_k = \sum_{i=1}^{k} dim(\lambda_i)$ and $d_0 = 0$. If $\lambda_m < B_0(x) < \lambda_{m+1}$, $\lambda_n < B_0(x) < \lambda_{n+1}$, $\forall x \in \Omega$, we have

$$i(B_0) = d_m,\ v(B_0) = 0,\ i(B_\infty) = d_n,\ v(B_\infty) = 0.$$

And Eq. (1.1) has at least $|d_m - d_n|$ pairs of nontrivial solutions.

This paper is organized as follows. In Section 2, we first introduce a variational setting for (1.2) and show the associated Euler-Lagrange functional. Then we construct an index theory for the fractional Schrödinger equations. Finally, in Section 3, we give the proof of Theorem 1.1.

PRELIMILARIES

Variational Settings

We first recall some definitions on the spaces X and X_0. In the sequel we set $Q = \mathbb{R}^{2N}\backslash\Lambda$ where

$$\Lambda = (\mathcal{C}\Omega) \times (\mathcal{C}\Omega) \subset \mathbb{R}^{2N},$$

and $\mathcal{C}\Omega = \mathbb{R}^N\backslash\Omega$. The space X is endowed with the norm defined as

$$\|u\|_X = \|u\|_{L^2(\Omega)} + (\int_Q \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dxdy)^{\frac{1}{2}}$$

And

$$X_0 = \{u \in X : u = 0 \ a.e. \ in \ \mathbb{R}^N\backslash\Omega\}. \tag{2.1}$$

Throughout this paper we use the following norm

$$\|u\|_{X_0} = (\int_Q \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dxdy)^{\frac{1}{2}} \tag{2.2}$$

which is equivalent to the usual one defined as in (2.1) (see [29]).

From [27] [30], we have

Lemma 2.1. *X_0 embeds continuously into $L^p(\Omega)$ for all $p \in [1, \frac{2N}{N-2s}]$, and compactly into $L^p(\Omega)$ for all $p \in [1, \frac{2N}{N-2s})$.*

On X_0 we define the functional

$$I(u) = \frac{1}{2}\int_Q \frac{(u(x)-u(y))^2}{|x-y|^{N+2s}}\,dxdy - \frac{1}{2}\int_\Omega F(x,u)\,dx. \tag{2.3}$$

Our hypotheses on f imply that f verifies the subcritical growth condition:

$$f(x,u) \le a_0(1+|u|^{p-1}) \text{ for a.e. } x \in \Omega, \forall u \in \mathbb{R}, (p \in (1, \frac{2N}{N-2s})).$$

And a standard argument shows that $I \in C^1(X_0, \mathbb{R})$ and critical points of I are solutions of (1.1).

Index Theory for Linear Fractional Laplacian Equation

In this subsection, we will establish an index theory to classify the following linear Laplacian Schrödinger equation

$$(-\Delta)^s u - B(x)u = 0 \text{ in } \Omega, \tag{2.4}$$

$$u = 0 \text{ in } \mathbb{R}^N \backslash \Omega.$$

For any $u, v \in X_0$, we define the following bilinear form

$$q_B(u,v) = \frac{1}{2}\int_Q \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}dxdy - \frac{1}{2}\int_\Omega B(x)u(x)v(x)\,dx. \tag{2.5}$$

For any $u, v \in X_0$, if $q_B(u,v) = 0$, we say that u and v are q_B-orthogonal. For any two subspaces X_1 and X_2, if $q_B(u,v) = 0$ for any $u \in X_1$, $v \in X_2$, we say that X_1 and X_2 are q_B-orthogonal.

Lemma 2.2. *For any function $B(x)$ continuous in $\bar{\Omega}$, there is a splitting*

$$X_0 = X^+(B) \oplus X^0(B) \oplus X^-(B) \tag{2.6}$$

such that

(a) $X^+(B), X^0(B)$, *and* $X^-(B)$ *are* q_B-*orthogonal*.
(b) $q_B(u,u) > 0, \forall u \in X^+(B)\backslash\{0\}$.
(c) $q_B(u,u) = 0, \forall u \in X^0(B)$.
(d) $q_B(u,u) < 0, \forall u \in X^-(B)\backslash\{0\}$.
(e) $X^0(B)$ *and* $X^-(B)$ *are finite dimensional*.

Proof. Let $l \in \mathbb{R}$ satisfies $l > B$ and $l > 0$. Then

$$(u,v)_{l,B} = 2q_B(u,v) + l(u,v)_{L^2(\Omega)} = \int_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}dxdy + \int_\Omega (l - B(x))u(x)v(x)\,dx,$$

also defines a Hilbert space structure. By Riesz representation theorem, there exists an operator $\mathcal{K}_l$ from X_0 to itself such that

$$(u,v)_{L^2(\Omega)} = (\mathcal{K}_l u, v)_{l,B}. \tag{2.7}$$

It follows from the fact that X_0 embeds compactly into $L^2(\Omega)$ that the operator $\mathcal{K}_l$ is a compact operator. Moreover, it is easy to check that $\mathcal{K}_l$ is a self-adjoint operator. Applying the spectral theory of compact self-adjoint operators on a Hilbert space, we know that there is a basis $\{e_j\}_{j\in N}$ of X_0, and a sequence $\mu_j \to 0$ in $\mathbb{R}$ as $j \to +\infty$ such that

$$(e_j, e_i)_{l,B} = \delta_{ij}, \mathcal{K}_l e_j = \mu_j e_j. \tag{2.8}$$

Therefore, for any $u = \sum_{j=1}^{\infty} c_j e_j$ and $v = \sum_{j=1}^{\infty} d_j e_j$, by the definition of formula (2.5), we have

$$q_B(u,v) = \frac{1}{2}(u,v)_{l,B} - \frac{l}{2}(u,v)_{L^2(\Omega)} = \frac{1}{2}((1-l\mathcal{K}_l)u,v)_{l,B} = \frac{1}{2}\sum_{j=1}^{\infty}(1-l\mu_j)c_j d_j,$$

And

$$q_B(u,v) = \frac{1}{2}\sum_{j=1}^{\infty}(1-l\mu_j)\,c_j^2.$$

Since $\mu_j \to 0$ as $j \to +\infty$, all the coefficients $(1-l\mu_j)$ are positive except a finite number. Then the result follows by

$$X^+(B) = \{x = \sum c_j e_j \,|c_j = 0 \; if \; 1-l\mu_j \le 0\},$$
$$X^0(B) = \{x = \sum c_j e_j \,|c_j = 0 \; if \; 1-l\mu_j \ne 0\},$$
$$X^-(B) = \{x = \sum c_j e_j \,|c_j = 0 \; if \; 1-l\mu_j \ge 0\}.$$

Definition 2.1. *For any function $B(x)$ continuous in $\overline{\Omega}$, we define*

$$i(B) = dim X^-(B), \;\; v(B) = dim X^0(B).$$

Lemma 2.3. *For any function $B(x)$ continuous in $\overline{\Omega}$, $v(B)$ is the number of linearly independent solutions of the following problem*

$$(-\Delta)^s u - B(x)u = 0 \; in \; \Omega,$$

$$u = 0 \; in \; \mathbb{R}^N \backslash \Omega.$$

In other words, $v(B) = dim\, ker((-\Delta)^s - B)$.

Proof. For any $u \in X_0(B)$, by the definition of $X_0(B)$ and $\mathcal{K}_l$ as in Lemma 2.2, u can be represented as $u = \sum_{1-l\mu_j=0} c_j e_j$ and $\mathcal{K}_l u = \sum_{1-l\mu_j=0} c_j \mu_j e_j$. Then $\mathcal{K}_l u - \frac{1}{l}u = \sum_{1-l\mu_j=0}(\mu_j - \frac{1}{l})c_j e_j = 0$. It follows that

$$l(u,v)_{L^2(\Omega)} = (l\mathcal{K}_l u, v)_{l,B} = (u,v)_{l,B}.$$

According to the definition of q_B, we obtain $q_B(u,v) = 0$, for any $v \in X_0$. This yields

$$(-\Delta)^s u - B(x)u = 0.$$

Recall that for a bounded symmetric bilinear form ψ defined on a Hilbert space X, its Morse index and nullity are defined as $m^-(\psi) = \max\{\dim(X_1): X_1 \text{ is a subspace of } X \text{ such that } \psi(x,x) < 0 \text{ for any } x \in X_1 \backslash \{0\}\}$.

Lemma 2.4. *For any function $B(x)$ continuous in $\overline{\Omega}$, $i(B)$ is the Morse index of q_B and $i(B) = \sum_{\lambda<0} v(B+\lambda)$.*

Proof. Assume that $q_B(u,u) < 0$ for any $X_1\backslash\{0\}$ with $\dim X_1 = k$. Let $e_1, \cdots, e_k$ be a basis of X_1. From Lemma 2.2, we obtain

$$e_j = e_j^+ + e_j^0 + e_j^-, e_j^* \in X^*(B), * = +, 0, -.$$

Then, it is suffices to show that $e_1^-, \cdots, e_k^-$ are linear independent. Now argue by contradiction, we assume that there exist not all zero numbers c_j such that $\sum_{j=1}^k c_j e_j^- = 0$.

On the one hand, $e = \sum_{j=1}^k c_j e_j = \sum_{j=1}^k c_j (e_j^+ + e_j^-)$ and $q_B(e,e) \geq 0$; on the other hand, $0 \neq e \in X_1$ and $q_B(e,e) < 0$. This is a contradiction.

In addition, since $q_B(u,u) < 0$ for any $u \in X^-(B)\backslash\{0\}$, we have $i(B) \leq m^-(q_B)$. Thus, $i(B)$ is the Morse index of q_B. For any $u \in X^0(B+\lambda)\backslash\{0\}$ with $\lambda < 0$, we have

$$q_B(u,v) - \frac{1}{2}\lambda(u,v)_{L^2(\Omega)} = 0, \forall v \in X^0.$$

By the definition of the operator $\mathcal{K}_l$, we have $\mathcal{K}_l u = \frac{1}{\lambda + l} u$. Clearly, u is an eigenvalue of $\mathcal{K}_l$ with eigenvalue $\frac{1}{\lambda+l}$. In addition, $1 - l\mu_j = \lambda\mu_j < 0$, we have $u \in X^-(B)$. This yields

$$\sum_{\lambda<0} X^0(B+\lambda) \subset X^-(B). \tag{2.9}$$

Moreover, since $\mu_j \to 0$ and $\mu_j > 0$, there a finite number of eigenvalues of $\mathcal{K}_l$ that satisfy $\mu_j > \frac{1}{l}$. Denote them by $r_1 \geq r_2 \cdots \geq r_k \geq \frac{1}{l}$ with the eigenfunction $e_1, e_2, \cdots, e_k$. For any $x \in X^-(B)$, x can be expressed as

$$x = \sum_{i=1}^k \sum_{\mu_j = r_j} c_j e_j = x_1 + x_2 + \cdots x_k.$$

Since $\mathcal{K}_l x_i = r_i x_i$, we obtain

$$(l\mathcal{K}_l x_i, v)_{l,B} = (r_i x_i, v)_{l,B} \text{ and } \frac{1}{r_i}(x_i, v)_{L^2(\Omega)} = (x_i, v)_{l,B}.$$

This yield

$$q_B(x_i, v) + (l - \frac{1}{r_i})(x_i, v)_{L^2(\Omega)} = 0.$$

So that $x_i \in X^0(B - (l - \frac{1}{r_i}))$. Hence, by (2.9), we have

$$X^-(B) = \oplus_{i=1}^k X^0(B - (l - \frac{1}{r_i})),$$

which completes the proof.

For any functions $B_1(x)$, $B_2(x)$ continuous in Ω, we write $B_1 \leq B_2$ if $B_1(x) \leq B_2(x)$ a.e. on Ω, and write $B_1 < B_2$ if $B_1 \leq B_2$ and $B_1(x) \leq B_2(x)$ on a subset of Ω with nonzero measure.

Lemma 2.5. *For any functions $B_1(x)$, $B_2(x)$ continuous in $\overline{\Omega}$ with $B_2 > B_1$, we have*

$$i(B_2) - i(B_1) = \sum_{s\in[0,1)} v(B_1 + s(B_2 - B_1)).$$

Proof. First we claim that there holds $i(B_1) + v(B_1) \leq i(B_2)$.

$$\text{In fact, for any } u \in (X^0(B_1) \oplus X^-(B_1))\backslash\{0\},$$

we have

$$q_{B_2}(u,u) = q_{B_1}(u,u) - \frac{1}{2}((B_2 - B_1)u,u)_{L^2(\Omega)} \leq -\frac{1}{2}((B_2 - B_1)u,u)_{L^2(\Omega)} < 0.$$

This yields $i(B_1) + v(B_1) \leq i(B_2)$ by Lemma 2.4.

Secondly, set $B(s) = B_1 + s(B_2 - B_1)$. We write $i(s) = i(B(s))$ and $v(s) = v(B(s))$ for short. For any $s \in [0,1)$, there holds $i(s + 0) = i(s) + v(s)$.

Let $i(s + 0) = k$. Since the index function is integer-valued, there exists $s' > s$ such that $i(p) = i(s + 0) = k, v(p) = 0, \forall p \in (s, s')$.

Arguing as in Lemma 2.2, we know that, there exist $e_1^p, \cdots, e_k^p \in X_0$, a sequence compact self-adjoint operator $\mathcal{K}_l^p$, and a sequence $\mu_j^p \to 0$ in $\mathbb{R}$ as $j \to +\infty$ such that

$$(e_j^p, e_i^p)_{l,B(s)} = \delta_{ij}, \tag{2.10}$$

and $\forall v \in X_0$, $1 \leq j \leq k$,

$$(e_j^p, v)_{L^2(\Omega)} = (\mathcal{K}_l^p e_j^p, v)_{l,B(s)} = \mu_j^p (e_j^p, v)_{l,B(s)}.$$

Moreover,

$$1 - l\mu_j^p < 0, 1 \leq j \leq k.$$

This yields,

$$(e_j^p, v)_{X_0} + ((l - B(p) - \frac{1}{\mu_j^p})e_j^p, v)_{L^2(\Omega)} = 0, \forall v \in X_0. \tag{2.11}$$

Fix $1 \leq j \leq k$. As we know, e_j^p and μ_j^p are bounded with $\mu_j^p > \frac{1}{l}$. Then, up to a subsequence, we assume that

$$e_j^p \rightharpoonup e_j \text{ in } X_0, e_j^p \to e_j \text{ in } L^2(\Omega) \text{ and } \mu_j^p \to \mu_j \text{ in } \mathbb{R}, \tag{2.12}$$

as $p \to s+0$. Thus, by (2.11), we obtain

$$(e_j, v)_{X_0} + ((l - B(s) - \frac{1}{\mu_j})e_j, v)_{L^2(\Omega)} = 0, \forall v \in X_0.$$

In addition, since $1 - l\mu_j^p < 0$, we obtain $1 - l\mu_j \leq 0$. It follows that

$$q_{B(s)}(e_j, e_j) = -\frac{1}{2}((l - \frac{1}{\mu_j})e_j, e_j)_{L^2(\Omega)} \leq 0.$$

So that

$$e_j \in X^0(B(s)) \oplus X^-(B(s)), 1 \leq j \leq k.$$

Moreover, by (2.10), we have $(e_j, e_i)_{L^2(\Omega)} = 0$ and $\mu_j^p = \left\|e_j^p\right\|_{L^2(\Omega)} \to \left\|e_j\right\|_{L^2(\Omega)} = \mu_j \geq \frac{1}{l}$. Thus, $e_1, \cdots, e_k$ are linear independent and

$$i(s) + v(s) \geq k = i(s+0).$$

The converse inequality holds by the first step. Therefore, $i(s) + v(s) = i(s+0)$.

In conclusion, we observe that the index function $i(s)$ is integer-valued, right continuous and non-decreasing on [0,1). Moreover, it is easy to verify that the function $s \to i(s)$ is left continuous, i.e. $i(s) = i(s-0)$. Then the value of $i(s) - i(B_1)$ must be equal to the sum of the jumps it experieces $[0, s)$. By the second step, we deduce that $i(B_2) - i(B_1) = \sum_{s\in[0,1)} v(B_1 + s(B_2 - B_1))$.

From Lemma 2.5, it is easy to deduce that:

Lemma 2.6. *For any function $B(x)$ continuous in $\overline{\Omega}$ there exists $\epsilon_0 > 0$ such that for any $\epsilon \in (0, \epsilon_0]$, we have*

$v(B+\epsilon) = 0 = v(B-\epsilon)$,
$i(B-\epsilon) = i(B), i(B+\epsilon) = i(B) + v(B)$.
In addition, if $v(B) = 0$, for any $\epsilon \in [0, \epsilon_0)$, we have
$v(B+\epsilon) = 0 = v(B-\epsilon)$,
$i(B-\epsilon) = i(B) = i(B+\epsilon)$.

PROOF OF THEOREM 1.1

In order to study the critical points of I, we will use the symmetric Mountain-Pass Theorem (see [1] or [32], Theorem 6.3] and a version of Clark's Theorem (see [25] or [15], Theorem 4.1]. Recall that $(u_n) \in X^0$ is a Palais-Smale (or (PS) for short) sequence for I, if $I(u_n)$ is bounded

and $I'(u_n) \to 0$. I satisfies the (PS)-condition if any such sequence contains a convergent subsequence.

Theorem 3.1. *Let $\Phi \in C^1(X, \mathbb{R}^1)$ be an even functional on a Banach space X. Assume $\Phi(0) = 0$ and Φ satisfies the (PS)- condition. If*

(Φ_1) *there exists a $E_1 \subset X$, $dim E_1 = k_1$ and $\rho > 0$ such that*

$$sup_{u \in E_1 \cap S_\rho} \Phi(u) < 0.$$

(Φ_2) *there exists a $E_2 \subset X$, $codim E_2 = k_2 < k_1$ such that*

$$inf_{u \in E_2} \Phi(u) > -\infty,$$

then Φ has at least $k_1 - k_2$ pairs of critical points with negative critical values.

Or, if

(Φ_3) *there exists a $G_1 \subset X$, $dim G_1 = l_1$ and $R > 0$ such that*

$$sup_{u \in G_1, \|u\| \geq R} \Phi(u) \leq 0.$$

(Φ_4) *there exists a $G_2 \subset X$, $codim G_2 = l_2 < l_1$ and $\rho > 0$ such that*

$$inf_{u \in G_2 \cap S_\rho} \Phi(u) > 0.$$

then Φ has at least $l_1 - l_2$ pairs of critical points with positive critical values.

Lemma 3.1. *Assume (f_0) and (f_∞) are satisfied. Then there exist $\epsilon, \rho > 0$ such that*

(1) $sup_{X^-(B_0 - \epsilon) \cap \partial \mathcal{B}_\rho} I < 0$,

(2) $inf_{X^+(B_0 + \epsilon) \cap \partial \mathcal{B}_\rho} I > 0$.

Proof. Set $F_1(x, u) = \int_0^1 f_1(x, \theta u)\, d\theta u$. Fix any $2 < p < 2^*$. By conditions (f_0) and (f_∞) , we have

$$f_1(x, u) = (B_\infty(x) - B_0(x))u + f_2(x, u).$$

Recall that, B_0, $B_\infty \in C(\overline{\Omega})$. Then, for any $\epsilon > 0$, there exists a $C_\epsilon > 0$, such that

$$|f_1(x, u)| \leq \epsilon |u| + C_\epsilon |u|^{p-1}, \forall (x, u) \in \Omega \times \mathbb{R},$$

and

$$|F_1(x, u)| \leq \frac{\epsilon}{2} |u|^2 + \frac{C_\epsilon}{p} |u|^p, \forall (x, u) \in \Omega \times \mathbb{R}.$$

Therefore,

$$-\frac{\epsilon}{2}\|u\|^2_{L^2(\Omega)}-\frac{C_\epsilon}{p}\|u\|^2_{L^p(\Omega)}\le\int_\Omega F_1(x,u)\,dx\le\frac{\epsilon}{2}\|u\|^2_{L^2(\Omega)}+\frac{C_\epsilon}{p}\|u\|^2_{L^p(\Omega)}.$$

Since X_0 embeds into $L^p(\Omega)$ compactly, there exists C^* which depends on Ω and p, such that

$$-\frac{\epsilon}{2}\|u\|^2_{L^2(\Omega)}-C^*\|u\|^p_{X_0}\le\int_\Omega F_1(x,u)\,dx\le\frac{\epsilon}{2}\|u\|^2_{L^2(\Omega)}+C^*\|u\|^p_{X_0}.$$

This yields, for any $u\in X_0$

$$I(u)\ge\frac{1}{2}\|u\|^2_{X_0}-\frac{1}{2}((B_0+\epsilon)u,u)_{L^2(\Omega)}-C^*\|u\|^p_{X_0}=q_{B_0+\epsilon}(u,u)-C^*\|u\|^p_{X_0},$$

and

$$I(u)\le\frac{1}{2}\|u\|^2_{X_0}-\frac{1}{2}((B_0-\epsilon)u,u)_{L^2(\Omega)}+C^*\|u\|^p_{X_0}=q_{B_0-\epsilon}(u,u)+C^*\|u\|^p_{X_0},$$

Noticing that $(q_{B_0+\epsilon}(u,u))^{\frac{1}{2}}$ and $((-q_{B_0-\epsilon}(u,u))^{\frac{1}{2}}$ are equivalent norm on $X^+(B^0+\epsilon)$ and $X^-(B_0-\epsilon)$, respectively. There exist constants $C_1, C_2>0$ such that

$$q_{B_0+\epsilon}(u,u)\ge C_1\|u\|^2_{X_0},\ \forall u\in X^+(B^0+\epsilon),$$

and

$$q_{B_0-\epsilon}(u,u)\le -C_2\|u\|^2_{X_0},\ \forall u\in X^-(B^0-\epsilon).$$

Hence, $\forall u\in X^+(B_0+\epsilon)$,

$$I(u)\ge C_1\|u\|^2_{X_0}-C^*\|u\|^p_{X_0}=(C_1-C^*\|u\|^{p-2}_{X_0})\|u\|^2_{X_0},$$

and $\forall u\in X^-(B_0-\epsilon)$

$$I(u)\le -C_2\|u\|^2_{X_0}+C^*\|u\|^p_{X_0}=(-C_2+C^*\|u\|^{p-2}_{X_0})\|u\|^2_{X_0}.$$

Now the lemma follows by choosing $\rho=\|u\|_{X_0}$ to be sufficiently small.

Lemma 3.2. *There exists $R>0$ such that*

$$sup_{X^-(B_\infty)\cap\mathcal{B}_R^c}I<0. \tag{3.1}$$

Proof. In fact, since $X^-(B_\infty)$ is a finite dimensional subspace, it is sufficient to show that, for any $u\in X^-(B_\infty)\backslash\{0\}$,

$$I(tu)\to-\infty,\text{ as }|t|\to\infty.$$

Assume that $F_2(x,u)=\int_0^u f_2(x,s)\,ds$. By (f_∞), we have $\frac{F_2(x,tu)}{t^2}\to 0$ a.e. on Ω. We deduce that

$$\lim_{|t|\to\infty}\int_\Omega \frac{F_2(x,tu)}{t^2}dx=0. \tag{3.2}$$

Thus, for any $u\in X^-(B_\infty)\backslash\{0\}$,

$$\lim_{|t|\to\infty}\frac{I(tu)}{t^2}=\frac{1}{2}\|u\|_{X_0}^2-\frac{1}{2}(B_\infty(x)u,u)_{L^2(\Omega)}+\lim_{|t|\to\infty}\int_\Omega\frac{F_2(x,tu)}{t^2}dx=q_{B_\infty}(u,u)<0.$$

Hence, we obtain the result.

Lemma 3.3. *There holds*

$$inf_{u\in X^+(B_\infty)}I(u)>-\infty.$$

Proof. It is sufficient to show that $I(u)\to+\infty$ as $u\in X^+(B_\infty)$, $\|u\|_{X_0}\to+\infty$. Arguing indirectly we assume that for some sequence $\{u_j\}\subset X^+(B_\infty)$, with $\|u\|_{X_0}\to+\infty$, there is $a>0$ such that $I(u_j)\le a$, for all j. Set $w_j=u_j/\|u_j\|_{X_0}$, we have $\|w_j\|_{X_0}=1$, $w_j\rightharpoonup w$ in $X^+(B_\infty)$, and $w_j\to w$ in $L^2(\Omega)$ and

$$\frac{a}{\|u\|_{X_0}^2}\ge\frac{I(u_j)}{\|u\|_{X_0}^2}=\frac{1}{2}\|w_j\|_{X_0}^2-\frac{1}{2}(B_\infty w_j,w_j)_{L^2(\Omega)}-\int_\Omega\frac{F_2(x,u_j)}{\|u_j\|_{X_0}^2}dx=q_{B_\infty}(w_j,w_j)-\int_\Omega\frac{F_2(x,u_j)}{\|u_j\|_{X_0}^2}dx\,. \tag{3.3}$$

Noticing that $w_j\to w$ in $L^2(\Omega)$, we have

$$\int_\Omega\frac{F_2(x,u_j)}{\|u_j\|_{X_0}^2}dx=\int_\Omega\frac{|F_2(x,u_j)|}{|u_j|^2}|w_j|^2dx\le 2\int_\Omega\frac{|F_2(x,u_j)|}{|u_j|^2}|w_j-w|^2dx+2\int_\Omega\frac{|F_2(x,u_j)|}{|u_j|^2}|w|^2dx=o(1). \tag{3.4}$$

From Fatou Lemma, we obtain

$$o(1)\ge q_{B_\infty}(w_j,w_j)\ge q_{B_\infty}(w,w).$$

Since $w\in X^+(B_\infty)$, we deduce that $w=0$ and $w_j\to 0$ in $L^2(\Omega)$. From (3.3) and (3.4), we have $o(1)=\|w_j\|_{X_0}^2$ and $w_j\to 0$ in X_0, which contradicts to the fact that $\|w_j\|_{X_0}^2=1$.

Lemma 3.4. *Any PS-sequence of the functional I is bounded.*

Proof. Let $\{u_j\}\subset X_0$ be such that $I(u_j)\to c$ and $I'(u_j)\to 0$. Now argue by contradiction. We assume that, up to a subsequence, $\|u_j\|_{X_0}\to\infty$ and set $v_j=\frac{u_j}{\|u_j\|_{X_0}}$, then $\|v_j\|_{X_0}=1$. Without

loss of generality, we assume that $v_j \rightharpoonup v$ in X_0, $v_j \to v$ in $L^2(\Omega)$ and that $v_j(x) \to v(x)$ a.e. on Ω. Moreover, $u_n(x) \to \infty$ if $v(x) \neq 0$. By conditions (f_0) and (f_∞), there exists constant $C_1 > 0$ such that

$$\frac{|f_2(x,u)|}{|u|} \leq C_1.$$

Then for any $\psi \in C_0^\infty(\Omega)$, we have

$$o(1) = \left(\frac{I'(u_j)}{\|u_j\|_{X_0}}, \psi\right)_{X_0} = (v_j, \psi)_{X_0} - (B_\infty v_j, \psi)_{L^2(\Omega)} - \int_\Omega \frac{f_2(x,u_j)}{u_j} v_j \psi dx \to (v, \psi)_{X_0} - (B_\infty v, \psi)_{L^2(\Omega)}.$$

This implies that

$$(-\Delta)^s v - B_\infty v = 0.$$

As we know $v(B_\infty) = 0$, we have $v = 0$. Thus, $v_j \rightharpoonup 0$ in X_0, $v_j \to 0$ in $L^2(\Omega)$. Hence, we get

$$o(1) = \left(\frac{I'(u_j)}{\|u_j\|_{X_0}}, v_j\right)_{X_0} = (v_j, v_j)_{X_0} - (B_\infty v_j, v_j)_{L^2(\Omega)} - \int_\Omega \frac{f_2(x,u_j)}{u_j} v_j^2 \, dx = \|v_j\|_{X_0}^2 + o(1).$$

This is a contradiction to the fact that $\|v_j\|_{X_0} = 1$.

Lemma 3.5. *Any PS-sequence of the functional I has a convergence subsequence.*

Proof. By Lemma 3.4, the (PS)-sequence $\{u_j\}$ is bounded. Assume that, up to a subsequence, $u_j \rightharpoonup u$ in X_0. Set $w_j = u_j - u$, then $w_j \rightharpoonup 0$ in X_0 and $w_j \to 0$ in $L^2(\Omega)$. In order to establish strong convergence, it suffices to show that $\|w_j\|_{X_0} \to 0$. Similar to Lemma 3.4, we have

$$o(1) = (I'(u_j), w_j)_{X_0} = (u_j, w_j)_{X_0} - (B_\infty u_j, w_j)_{L^2(\Omega)} - \int_\Omega \frac{f_2(x,u_j)}{u_j} u_j w_j \, dx = \|w_j\|_{X_0}^2 + o(1),$$

which implies $w_j \to 0$ in X_0 and completes the proof.

Proof of Theorem 1.1

Case $i(B_0) > i(B_\infty)$. Condition (Φ_1) of Theorem 3.1 verified by (1) of Lemma 3.1 with $E_1 = X^-(B_0 - \epsilon)$. For sufficiently small ϵ, by lemma 2.6, we have $\dim E_1 = i(B_0 - \epsilon) = i(B_0)$. In addition, Lemma 3.3 implies I satisfies condition (Φ_2) of Theorem 3.1 with $E_2 = \dim X^+(B_\infty)$ and $\operatorname{codim} E_2 = i(B_\infty)$. Hence, I has at least $i(B_0) - i(B_\infty)$ pairs of nontrivial solutions.

Case $i(B_\infty) > i(B_0)$. Lemma 3.2 says that I satisfies (Φ_3) with $\dim G_1 = \dim X^-(B_\infty) = i(B_\infty)$. With $G_2 = X^+(B_0 + \epsilon)$, the condition (Φ_4) holds by (2) of Lemma 3.1. Moreover, by Lemma 2.6,

$\mathrm{codim}G_2 = i(B_0)$ for ϵ small. This yield I has at least $i(B_\infty) - i(B_0)$ pairs of nontrivial solutions. In conclusion, I has at least $|i(B_0) - i(B_\infty)|$ pairs of nontrivial solutions.

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