

Knotted Configurations in the Continuous Heisenberg Spin Chain with Lower Bound of the Energy

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ABSTRACT

When studying the classical Heisenberg spin models, we usually map the normalised unit vector, that represents the spin, on the tangent of a space curve. The total chirality of the curve (the spin configuration) i.e., the total momentum of the spin system, is a conserved quantity [1] and therefore self-crossing of the curve is not allowed. Using this fact and the non-contractibility of some closed loops in $SO(3)$, we have proposed new topological spin configurations for the Heisenberg spin model [1]. Now we propose new topological spin configuration for the Heisenberg model, which represents a knot and has higher lower bound for the energy as the previously proposed configurations. The same analysis holds for a thin elastic rod that is bent to an open knot. PACS numbers: 75.10. Pq, 75.10. Hk, 02.40. Hw, 02.10. Kn

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INTRODUCTION

In a recent paper [1] we have proposed new topological configurations for the one-dimensional continuous Heisenberg spin model which have lower bound for the energy. In a following paper [2] these configurations have been constructed as solutions of the Landau-Lifshitz equation ($n_t = n \times n_{ss}$) where subscripts denote derivatives with respect of the arc length s and time t . Our analysis is based on the classical theory of curves and the classical Hamiltonian mechanics of the spin chain. The order parameter is normalised vector n (the spin) where $n^2 = 1$. In angle variables $n = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$. In these variables the Hamiltonian reads:

$$H = J \int_{-L}^{+L} (\theta_s^2 + \sin^2 \theta \phi_s^2) ds \quad (1)$$

where the subscript s stands for d/ds and s denotes the coordinate along R^1 (the system is one-dimensional). The classical mechanics for this system has been worked out by Tjon and Wright [3]. They have realised that ϕ and $\cos \theta$ can be considered as conjugated generalised coordinate and momentum in which case the Poisson bracket reads $[\phi(x), \cos \theta(y)] = \delta(x - y)$. Then the generator of translations (linear momentum) is:

$$P = \int_{-L}^{+L} (1 - \cos \theta) \phi_s ds \quad (2)$$

It turns out that the linear momentum P is a constant of the motion. In order to use the properties of space curves we will map the unit vector n to the unit tangent of a space curve [4], so different space curves will represent different spin configurations. We will use homogeneous boundary conditions where spins at $\pm L$ will be parallel. Then curves will tend to the straight line as $s \rightarrow \pm L$. We are going to use the properties of the writhe of a curve (which characterises the chirality of the curve). For a closed curve it is given by the following expression: [5]:

$$Wr = \frac{1}{4\pi} \oint ds \oint ds' \frac{(r(s)-r(s')) \cdot (n(s)-n(s'))}{|r(s)-r(s')|^3} \quad (3)$$

The tip of the radius vector r draws the curve, and n is its unit tangent. Using a theorem by Fuller [6] one can express Wr with respect to a reference curve C_0 [7]:

$$Wr = Wr_0 + \frac{1}{2\pi} \int_{-L}^{+L} \frac{n_0 \times n \cdot \frac{d}{ds}(n_0 + n)}{1 + n_0 \cdot n} ds \quad (4)$$

Here Wr_0 is the writhe of the reference curve and after appropriate choice of the reference curve we [7] get:

$$Wr = \frac{1}{2\pi} \int_{-L}^{+L} (1 - \cos\theta) \phi_s ds \quad (5)$$

RESULTS AND DISCUSSION

Following the analysis in [1] we note that the write Wr coincides with the total momentum P and hence the writhe $W r$ is conserved quantity. We note also that when one region of the curve crosses another one, the writhe $W r$ suffers discontinuity by 2. Let us analyse now possible knot configurations for the continuous one-dimensional classical Heisenberg spin chain. In [8] the authors present a knot solution for the nonlinear Schrödinger equation (NLSE). They map the NLSE onto the following equation for a space curve:

$$R_t = R_s \times R_{ss} \quad (6)$$

Where the tip of the vector $R(s, t)$ draws a non-stretching space curve. Here's represents the arc length of the space curve. Now let us take the derivative of the above equation with respect to the arc length s and note that $dR/ds = n$, where n represents the tangent to the space curve

$$R_{st} = R_{ss} \times R_{ss} + R_s \times R_{sss} = R_s \times R_{sss} \quad (7)$$

Here we recognise the Landau-Lifshits equation which turns out to be equivalent to the NLSE.[9]

$$n_t = n \times n_{ss} \quad (8)$$

n represents the normalised spin vector which has been mapped to the tangent of a space curve that now represents a spin configuration. Now we are ready to examine the energy of the spin configuration that represents a knot. It is well known that for a closed space curve [10]:

$$\oint k ds \geq 2\pi \quad (9)$$

We will use now yet another inequality for the integral curvature which concerns knots, namely the Fary-Milnor theorem [11]:

$$\oint k ds \geq 4\pi \quad (10)$$

The curvature is different from zero $k \neq 0$ only in the interval $s \in (-L, +L)$. Let us consider one such configuration representing an open knot: the space curve representing the knot goes from $-L$ to $+L$. In order to get a closed curve this is completed by a straight line between $-L$ and $-\infty$ and between $+L$ and $+\infty$ and is closed by a semi-circle at infinity. The straight segments and the semi-circle at infinity have curvature k zero. The writhe there is also 0. Then the above inequality becomes:

$$\int_{-L}^{+L} k ds \geq 4\pi \quad (11)$$

Let us note here that the Fary-Milnor inequality has been used to estimate the lower bound of the number of localised electronic states on a trefoil knot [12].

Now we will apply the following Cauchy-Schwarz inequality to the integral curvature:

$$\left(\int_{-L}^{+L} k ds \right)^2 \leq 2L \int_{-L}^{+L} k^2 ds \quad (12)$$

If we introduce Euler angles the curvature is given by the following expression:

$$k^2 = \theta_s^2 + \sin^2 \theta \phi_s^2 \quad [13]$$

We are ready to calculate the lower bound of the energy for the spin chain with a knot configuration. Now the energy of the spin chain satisfies the following inequality:

$$H = J \int_{-L}^{+L} (\theta_s^2 + \sin^2 \theta \phi_s^2) ds = J \int_{-L}^{+L} k^2 ds \geq \frac{J \left(\int_{-L}^{+L} k ds \right)^2}{2L} \geq J \frac{16\pi^2}{2L} = J \frac{8\pi^2}{L} \quad (13)$$

Here we note that the lower bound of the energy for the knot-configuration is 4 times higher than the energy for the 2π - twist configuration studied in [1].

The same analysis can be applied to the case of a thin elastic rod that is bent/twisted into an open knot. Its energy has the following lower bound (for the thin rod that is twisted by 2π and whose end are glued together, see [14]):

$$E = I_{min} \int_{-L}^{+L} k^2 ds \geq I_{min} \frac{16\pi^2}{2L} = I_{min} \frac{8\pi^2}{L} \quad (14)$$

Where I_{min} stands for the minimal bending/twisting rigidity of the rod. In this case too the lower bound of the energy of a knotted thin rod is 4 times higher than the lower bound for 2π twisted rod.

CONCLUSIONS

As in our previous analysis [1] we have used the fact that the writhe Wr , that coincides with the total momentum P is a conserved quantity. This does not allow two parts of the curve that represents the spin configuration to cross each other. Let us note that for an infinite chain the lower bound of the energy goes to 0. We have shown that there is a second kind of topological configuration (the knot) for the Heisenberg spin model in the one-dimensional case. This knot configuration has 4 times higher lower bound of the energy than the previous one. The same analysis is applicable to the case of a thin rod that is bent/twisted into a knot.

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