

Planar Gravitating Objects in General Relativity

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ABSTRACT

We consider the case of a planar gravitating object in General Relativity and find glaring inconsistencies in the Einstein Field Equations.

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INTRODUCTION

Consider an object (something approximating a black hole, if you will) distributed in the $x - y$ plane of constant density, ρ , for finite thickness, $-h < z < 0$. This corresponds to the case of a constant gravitational field in the z -direction in Newtonian gravity—the simple case you learned in High School where projectiles follow parabolic trajectories & escape velocity is ∞ —i.e., nothing can escape the gravity: a projectile shot with any velocity upward returns to earth. The mass per unit area is finite $= \rho h$; the gravitational field depends on this finite quantity and not the total mass of the plane, which is infinite—both in the relativistic & Newtonian case. However, there are easily calculable contradictions in the Einstein Field Equations [EFE], as we will see in this brief missive.

THE METRIC AND STRESS ENERGY TENSOR

I maintain that the correct way to proceed is to choose $G_{00} = c^2$ so that t is the reading on a clock rather than a mathematical variable of obscure physical meaning. But everyone is confused about that & this is not the place to set the record straight. Thus, here we will surrender to the usual procedure of choosing a diagonal four-metric, with G_{00} a function of z —since the gravitational field is clearly static and owing to the symmetry of the problem, G_{00} shouldn't depend on x, y or t .

Thus,

$$G_{\mu\nu} = \begin{bmatrix} -\delta_{mm} & 0 \\ 0 & c^2(1 + \frac{\varphi}{c^2})^2 \end{bmatrix}, G^{\mu\nu} = \begin{bmatrix} -\delta_{mm} & 0 \\ 0 & \frac{1}{G_{00}} \end{bmatrix}$$

φ is the gravitational potential. We choose the form for G_{00} on the basis of the fact that in an accelerating reference frame accelerating at constant acceleration, a , $G_{00} = c^2(1 + az/c^2)^2$ [1] and on the Principle of Equivalence between acceleration and gravity. However, this step has no actual content, being merely a transformation of the dependent variable ($G_{00} \rightarrow \varphi$).

The stress energy tensor for a perfect fluid in relativity is (it is standard practice to model neutron stars, black holes, &c. as perfect fluids [2]),

$$T^{\mu\nu} = -G^{\mu\nu}P + (\rho + \frac{P}{c^2})\delta_0^\mu\delta_0^\nu$$

where P is the pressure. See [2] for this expression: we reverse the sign of $G^{\mu\nu}$ since we are using the signature $(+, -, -, -)$ for $G_{\mu\nu}$, while [2], like most authors uses the signature $(-, +, +, +)$, but I like interpreting the line element straightforwardly as $cd\tau$ where $d\tau$ is the proper time interval.

Moreover, we have inserted δ_0^μ for the 4-velocity, V^μ , because the gravitating object is at rest, being assumed to be in static equilibrium.

The covariant stress energy tensor appearing in the EFE is,

$$T_{\mu\nu} = G_{\mu\rho}G_{\nu\sigma}T^{\rho\sigma} = -G_{\mu\nu}P + (\rho + P/c^2)G_{\mu 0}G_{\nu 0}.$$

THE CONNECTION COEFFICIENTS

The connection coefficients are easy to compute because $G^{\mu\nu}$ is diagonal and only $\frac{\partial G_{00}}{\partial z}$ is non-zero. Remember The only non-zero components are,

$$\Gamma_{\rho\sigma}^\mu = \frac{1}{2}G^{\mu\lambda}(\frac{\partial G_{\rho\lambda}}{\partial u_\sigma} + \frac{\partial G_{\sigma\lambda}}{\partial u_\rho} - \frac{\partial G_{\rho\sigma}}{\partial u_\lambda})$$

where the prime denotes differentiation with respect to z . It facilitates computation to note,

$$\Gamma_{r0}^0 = \frac{\varphi'\delta_{r3}}{c^2(1 + \frac{\varphi}{c^2})}, \Gamma_{00}^m = \varphi'(1 + \frac{\varphi}{c^2})\delta_{m3}$$

THE FIELD EQUATIONS

$$R_{\mu\nu} = \Gamma_{\mu\nu,\lambda}^\lambda - \Gamma_{\mu\lambda,v}^\lambda + \Gamma_{\mu\nu}^\lambda\Gamma_{\lambda K}^K - \Gamma_{\mu K}^\lambda\Gamma_{\nu\lambda}^K$$

where a comma denotes partial differentiation. This is easy to compute because there are so few non-zero connection coefficients. The results are (BTW: Greek indices range $(0,1,2,3)$; Latin $(1,2,3)$),

$$R_{00} = \varphi''\left(1 + \frac{\varphi}{c^2}\right), R_{0m} = 0, R_{mm} = \frac{-\varphi''}{c^2\left(1 + \frac{\varphi}{c^2}\right)}\delta_{m3}\delta_{n3}$$

$$R = G^{\mu\nu}R_{\mu\nu} = \frac{2\varphi''}{c^2\left(1 + \frac{\varphi}{c^2}\right)}.$$

The EFE are [2],

$$R_{\mu\nu} - \frac{1}{2} G_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu}.$$

$(\mu, \nu) = (1,1)$ or $(2,2)$ yields,

$$\frac{\varphi''}{1 + \frac{\varphi}{c^2}} = \frac{8\pi G P}{c^2},$$

$(\mu, \nu) = (3,3)$ yields,

$$0 = \frac{8\pi G P}{c^2},$$

$(\mu, \nu) = (0,0)$ yields,

$$0 = \frac{8\pi G}{c^2} \left(1 + \frac{\varphi}{c^2}\right)^2 \left[-P + \left(\rho + \frac{P}{c^2}\right)c^2 \left(1 + \frac{\varphi}{c^2}\right)^2\right]$$

The contraction of the EFE with $G^{\mu\nu}$ yields,

$$-R = \frac{8\pi G}{c^4} T,$$

or,

$$\frac{\varphi''}{1 + \frac{\varphi}{c^2}} = -\frac{4\pi G}{c^2} \left[-4P + \left(\rho + \frac{P}{c^2}\right)c^2 \left(1 + \frac{\varphi}{c^2}\right)^2\right].$$

The equations are severely self-contradictory, but if we go with the last equation (the contraction of the EFE), we obtain, in the Newtonian Limit ($c \rightarrow \infty$),

$$\varphi'' = -4\pi G \rho,$$

which has the wrong sign.

Though the EFE are hopelessly inconsistent & the whole theory can't be relied upon, some progress might be gained by writing the EFE in the form,

$$R_{\mu\nu} - \frac{1}{2} G_{\mu\nu} R = -\frac{8\pi G}{c^4} T_{\mu\nu}$$

the sign change being attributed to a poor understanding of the Newtonian Limit, which was used to compute the constant of proportionality, $+ 8\pi G / c^4$.

COSMIC EXPANSION

As an afterthought, note that this sign change has an effect on the Friedmann equations of relativistic cosmology (theory of cosmic expansion/Big Bang). Namely, the sign of $\ddot{a}$, where $a(t)$ is the cosmic scale factor is reversed, explaining why the expansion of the universe is accelerating ($\ddot{a} > 0$) without introducing the concept of dark energy or a cosmological constant. This may be progress. Moreover, the spatial curvature of the universe should be negative for

any values of $\ddot{a}$, $\dot{a}$ and P . The universe is spatially open & of infinite spatial extent. See [2] for details on the Friedmann cosmology.

CONCLUSION

We consider a very simple problem in relativistic gravity—the equivalent of a constant gravitational field in the Newtonian Limit—and find severe contradictions in the field equations in the region where there is matter, although not in the vacuum outside the gravitating object, suggesting that we have a good grasp of the theory of gravity in vacuum conditions, but that our theory of gravity in the presence of matter is flawed. Much work remains to be done to fill in the gap. The effort to construct a correct theory of gravity should be empirical rather than intellectual—let us never again be led down the garden path by another Einstein. We don't need creative geniuses to study this problem, but unimaginative laboratory workers. Theories of physics simply record the results of experiments like the notations chemists use—let us lose our ardor for “prediction” and “explanation.”

References

1. Misner, Thorne & Wheeler, Gravitation, Freeman Press, 1973
2. Guidry, Modern General Relativity, Cambridge University Press, 2019