



Quantum Theory and Idealism

Stuart Boehmer

Unaffiliated, Wichita, Kansas, United States

ABSTRACT

We find that the Wigner Probability Distribution has a fully classical interpretation, and that the microscopic world is not a "black box" (per the Copenhagen Interpretation). Probability represents our state of knowledge of a system. However, in quantum theory, as opposed to classical theory, cause and effect is at the level of our state of knowledge, $\rho(p, q)$, not the state of the world, (p, q) . Quantum theory is idealistic in the sense of Berkeley: the world is a conscious, shared hallucination.

Keywords: quantum theory, idealism, hard problem of consciousness, Bell inequality, Bertlmann's socks, EPR paradox

INTRODUCTION

If we define probability as a tool for calculating mean values (expectation values) of quantities (observables), $Q(p, q)$, rather than as the number of occurrences of something divided by the number of runs of the experiment, then negative probabilities are not problematic and quantum theory, interpreted in terms of the Wigner Probability Distribution (it is not a "quasi" probability distribution if we regard negative probabilities as acceptable), is fully classical: the probability distribution, $\rho(p, q)$, simply represents our state of knowledge of a system. The whole matter would have been settled in the 1930s with the publication of the papers of Dirac [1] and Heisenberg [2], if it were not for Wigner's [3] unfortunate discovery that the distribution sometimes takes negative values, obfuscating the matter for another 90 years, until today. The interpretation of quantum theory should not be controversial or problematic.

THE WIGNER PROBABILITY DISTRIBUTION

The probability matrix (thermodynamic average of $\bar{\psi}(q')\psi(q)$) is the Fourier Transform of the Wigner Distribution,

$$\hat{\rho}\left(q + \frac{y}{2}, q - \frac{y}{2}\right) = \int \rho(p, q) e^{-\frac{ipy}{\hbar}} d^{3N}p,$$

where p, q and y are $3N$ -dimensional vectors, where N is the number of particles in the system.

Usually, the Wigner Probability Distribution is defined as the Fourier Transform of the probability matrix, which is regarded as fundamental, the Wigner Distribution being a mere curiosity.

However, we regard the Wigner Distribution as fundamental and the probability matrix (or wave function) as merely a convenient mathematical transformation which facilitates the solution of difficult physical problems, which has no direct physical meaning in itself. The Schrödinger/Heisenberg equation for the probability matrix (or wave function) is much easier to solve than a direct solution of the quantum Liouville equation,

$$0 = \frac{\partial \rho}{\partial t} + \frac{\mathbf{p}}{m} \frac{\partial \rho}{\partial \mathbf{q}} - \sum_{n=0}^{\infty} \frac{(-1)^n (\hbar/2)^n}{(2n+1)!} \frac{\partial^{2n+1} V}{\partial \mathbf{q}^{2n+1}} \frac{\partial^{2n+1} \rho}{\partial \mathbf{p}^{2n+1}}. \quad (1)$$

In the classical limit ($\hbar \rightarrow 0$), this becomes the classical Liouville equation,

$$0 = \frac{\partial \rho}{\partial t} + \frac{\mathbf{p}}{m} \frac{\partial \rho}{\partial \mathbf{q}} - \frac{\partial V}{\partial \mathbf{q}} \frac{\partial \rho}{\partial \mathbf{p}}, \quad (2)$$

which has the solution,

$$\rho(\mathbf{p}, \mathbf{q}, t) = \rho(\mathbf{p} - \mathbf{p}_0(t), \mathbf{q} - \mathbf{q}_0(t), 0), \quad (3)$$

where $(\mathbf{p}_0(t), \mathbf{q}_0(t))$ are solutions of the classical Hamiltonian equations,

$$\dot{\mathbf{p}} = -\frac{\partial V}{\partial \mathbf{q}}, \dot{\mathbf{q}} = \frac{\mathbf{p}}{m}. \quad (4)$$

From (3), we see that, in classical systems, the individual particles are following trajectories, we just are unsure of exactly which. Cause and effect (or push and pull) are at the level of the state of the world, $(\mathbf{p}, \mathbf{q})$, and $\rho(\mathbf{p}, \mathbf{q})$ merely represents our state of knowledge.

In quantum theory, the situation is different. Equation (1) has no resolution such as (3). Push/pull/cause/effect are at the level of our state of knowledge!

This is perplexing: how *could* something as ethereal as the experimenter's frame of mind influence the outcome of events in the material world?

It is, however, easily understood in terms of the idealism of Berkeley, Hume et al.: the world is in the mind of conscious observers, and "matter" is a hallucination caused by memory and muscle and visual sensations. As Einstein once famously remarked, "The world is merely an illusion, albeit a very persistent one."

We could settle for studying the properties of the hallucination of matter *qua* a hallucination/illusion and many would say that that is all we can ever hope to do; however, I believe—as we will see in the Conclusion—that we can go further.

Hume was of the opinion that there is no such thing as cause and effect, but only correlations, and perhaps this further clarifies the solution to the mystery. "Cause" and "effect" are illusions caused by the muscle sensations of pushing and pulling. In defense of Hume's viewpoint, note this: if a correlation of *exactly* 1.0 could be established, we could say, *by definition*, cause and

effect were involved. But in any experiment, there is always a small amount of error, so it can never really be established empirically that a correlation is exactly 1.0.

THE MEASUREMENT PROBLEM

When we make a measurement, this instantly updates our state of knowledge, $\rho(\mathbf{p}, \mathbf{q})$.

This fully solves the measurement problem.

That there are two types of evolution of $\psi(\mathbf{q})$ or $\rho(\mathbf{p}, \mathbf{q})$, one for between measurements and one for during measurements [4] is no longer an enigma. The instantaneous collapse of the wave function is not faster-than-light propagation of some physical force, but merely the instant updating of our state of knowledge that occurs when a measurement has been made. There is no “spooky action at a distance.”

The measurement problem of quantum theory is no more enigmatic than the measurement problem of classical theory: the resolution is exactly the same.

THE WIGNER’S FRIEND PARADOX

The paradox of Wigner’s Friend [5] (and its variation: Schrödinger’s cat [6]) is easily resolved with the forgoing theory of measurement: -

Wigner and his friend have different information available to them, and, consequently, different $\rho(\mathbf{p}, \mathbf{q})$, different states of knowledge.

When the box is opened and Wigner and his friend have their states of knowledge equalized. Voilà!

EINSTEIN, PODOLSKY AND ROSEN AND BELL. [7,8,9,10]

The prevailing opinion today is that there is “spooky action at a distance” or something non-local/non-realistic about quantum theory. (I can accept that such things might exist in a non-materialistic reality; what I dispute is whether quantum theory in its present formulation documents this).

Certain correlations among measurements made at distant points in certain experiments are claimed to be counterintuitive. Bell [10] claims that these correlations are not of the trivial kind he calls “Bertlmann’s socks.” Heisenberg once famously declared that entanglement was the essential paradox of quantum theory.

Put a black marble in one box, a white marble in another, close the boxes, mix them up and remove them to a large spatial separation. If observer A opens one box and finds a white marble, then he instantly knows that B will find a black marble. Spooky action at a distance? Faster-than-light propagation of some pseudo-force?

But Bell [10] claims it is not this simple: something more enigmatic is somehow going on.

We can show in a few simple steps that he was wrong. (By the way, the error in Bell's argument [9] is the assumption that measurements made at separate points must be independent: but this begs the issue—they are not, as we shall show presently.)

If a two-particle probability distribution, $\rho(\mathbf{p}_1, \mathbf{q}_1, \mathbf{p}_2, \mathbf{q}_2)$ can be factored in the form $\rho'(\mathbf{p}_1, \mathbf{q}_1)\rho''(\mathbf{p}_2, \mathbf{q}_2)$, the two particles are statistically independent, otherwise they are correlated. "Entanglement," as we shall show presently, is a simple case of correlation: Bertlmann's socks.

The wave function (at some fixed instant of time) of the original EPR paradox [7,9] is, $\psi(q_1, q_2) = \delta(q_2 - q_1 - q_0)$, where q_0 is a parameter (the separation of the two particles at that instant) and δ is the Dirac Delta Function. A straightforward calculation shows at once that,

$$\rho(p_1, q_1, p_2, q_2) = \delta(q_2 - q_1 - q_0)\delta(p_2 + p_1),$$

which cannot be resolved in the form $\rho'(p_1, q_1)\rho''(p_2, q_2)$: entanglement = correlation, Bertlmann's socks! Voilà!

Similarly, for the Bell spin/polarization experiment [9], where the wave function is,

$$\psi(\mathbf{s}_1, \mathbf{s}_2) = \frac{1}{\sqrt{2}}(|\uparrow\rangle|\uparrow\rangle + |\downarrow\rangle|\downarrow\rangle),$$

where $|\uparrow\rangle$ is the spin-up state and $|\downarrow\rangle$ is the spin-down state and $\mathbf{s}_n$ is the spin of particle n , the resulting Wigner Probability Distribution, $\rho(\mathbf{s}_1, \mathbf{q}_1, \mathbf{s}_2, \mathbf{q}_2)$ cannot be resolved as $\rho'(\mathbf{s}_1, \mathbf{q}_1)\rho''(\mathbf{s}_2, \mathbf{q}_2)$ (see the Appendix): entanglement = correlation, Bertlmann's socks! Voilà!

As we stated in the Introduction, the whole matter would have been settled in the 1930s and be regarded as noncontroversial were it not for Wigner's unfortunate discovery, which obfuscated the issue for another 90 years, until today.

It is interesting to note that indistinguishable particles are just entangled/correlated distinguishable particles.

Thus, Bosons: -

$$\psi(1,2) = \frac{1}{\sqrt{2}}(\psi_1(1)\psi_2(2) + \psi_2(1)\psi_1(2)),$$

Fermions: -

$$\psi(1,2) = \frac{1}{\sqrt{2}}(\psi_1(1)\psi_2(2) - \psi_2(1)\psi_1(2)).$$

Thus, we could conceivably conduct a theoretical study of classical indistinguishable particles as a special case of correlation among the particles and the resulting thermodynamic statistics, though the result would probably be a mere curiosity.

THE COPENHAGEN INTERPRETATION

According to the well-known Copenhagen Interpretation, originally formulated by Bohr and Heisenberg in Copenhagen, we should not attempt to visualize the microscopic world. It is a “black box,” and we can investigate it only by use of macroscopic measuring instruments which are classical and visualizable.

But this is not true: there are robust quantum effects in the macroscopic and mesoscopic worlds; they all blend together and there is no clear cutoff between the various worlds. Vision is the result of the absorption of individual photons, as is the photoelectric effect and particle tracks can be seen through a not-too-high magnification microscope in a cloud chamber, among other things, not the least of which is the stability of matter itself: the original problem that quantum theory was invented to explain.

If the microscopic world is a black box, so are the mesoscopic and macroscopic worlds: neither can be comprehended or visualized.

However according to the results of the first two sections of this paper, there is no essential difference between classical and quantum probability distributions: the microscopic, mesoscopic and macroscopic worlds can be visualized in the traditional way. There is no mystery about what a “photon” is—the problem that Einstein famously remarked gave him more trouble than General Relativity is easily resolved: it is just a particle in the ordinary sense of the word, though the quantum theory is *essentially* indeterministic: the notion of probability is fundamentally irreducible, as opposed to the situation in classical statistical thermodynamics.

Maxwell once famously stated presciently that he believed that thermodynamic probabilities would prove to be irreducible and a fundamental part of nature. He apparently lacked clairvoyance regarding the astonishing consequences of such a circumstance, which we shall examine in section 8.

UNDEFINED STATES

The entanglement issue has been further obfuscated by the claim [10] that a quantum system is not in a definite state before measurement, that measurement alone determines the state of the system, that, before measurement, the system is in all states at once.

This misperception seems to arise from the assumption that the wave function—of some exotic physical significance—is fundamental and the negative probabilities of the Wigner Distribution are problematic.

If we accept the proposition that negative probabilities are fully acceptable—redefining the elementary theory’s definition of “probability”—then, just as in classical theory, $\rho(\mathbf{p}, \mathbf{q})$ is simply a representation of our state of knowledge within the confines of the uncertainty principle.

If, for instance, we measure p , $\rho(p, q)$ is changed from its current value to

$\delta(p - p')H_{\infty}(q - q')$, where (p', q') are the measured values and $H_{\Delta}(x)$ is the Heaviside function ($= \frac{1}{\Delta}$ when $x^2 < \frac{1}{4}\Delta^2$, 0 otherwise). This is a necessary consequence of the indeterminism of quantum theory. Measurement disturbs the system, but from this we cannot infer that the system was not in a definite state before the measurement.

From the idea that measurement alone determines the state of the system—which is false, as we have just seen—Bell [9,10] derives the principle that measurements made at distant points should be statistically independent, but they are not, as we have seen in section 5. The particles are correlated/entangled at the moment of creation/interaction—thus it is a simple case of a common history, despite claims otherwise [9]—and this correlation is preserved upon separation of the particles if further interaction with the rest of the universe in these extremely sensitive experiments can be avoided, preventing decoherence or destruction of the correlation.

CONCLUSION: THE HARD PROBLEM OF CONSCIOUSNESS AND THE HARD PROBLEM OF MATTER

The “hard problem of consciousness” is the question of how *can* electrochemical activity in a lump of meat result in something as alien as consciousness. (The “easy problem” is the mechanical question of *how* the brain works—hard enough, but conceptually solvable.)

The simple and straightforward answer is that it *can't*. The hard problem of consciousness is fundamentally unsolvable. If there is matter and the world is not dualistic, then we should all be mindless automatons without insight.

Writers such as D. C. Dennett [11] perceive this, but their ardor for materialism/atomism/reductionism/analysis deludes them into taking the absurd position that consciousness is a mere illusion. But what is experiencing this illusion if not a conscious being? And whom is the audience that Dennett writes for if we are all mindless automatons with whom it is impossible to communicate? Are we to understand that Dennett has no comprehension of his own thesis?

If we do not allow materialists to waffle, the statement that there should be no such thing as consciousness is a *firm* prediction of their theory. It is clearly false. Materialism is an incorrect theory of experience and must be abandoned. This confirms the conclusions of section 2 that quantum theory predicts that nature is essentially idealistic.

Materialism is dead. Atomism/reductionism/analysis are dead. The world is not built of atoms the way a house is built of bricks. The world is not “bottom up”/analytic but “top down”/synthetic. The macroscopic world is fundamental, and the microscopic world is a derivation or an inference, not the other way around. Nature is not a meaningless game of billiards with no object as militant atheists such as Steven Weinberg (who famously stated that, “The more the world seems comprehensible, the more it also seems meaningless.”) seem to think. Nature is neither comprehensible nor meaningless. Einstein once stated that, “The most incomprehensible thing about the world is that it is comprehensible;” but it is *not*

comprehensible: the materialistic picture of a swarm of atoms is just a *model* of a hallucination studied *qua* a hallucination (see section 2, above).

Enter: idealism. The hypothesis that consciousness is fundamental and that the seemingly substantial and solid material world is an evanescent hallucination created by anthropomorphic attributions of bodily sensations and memory to the outer world.

Voilà! At one stroke, we have solved the hard problem of consciousness: we postulate the existence of consciousness by *axiom*, the way that materialists postulate the existence of matter without attempting to explain what it is or where it comes from. Consciousness—not matter—is the fundamental “stuff” of experience.

A new problem now arises; let us christen it “the hard problem of matter.” Why does so much of the world seem to be publicly available as the illusion of a world filled with light and matter? Why isn’t the whole world an inner, private, subjective experience which follows no more regularity (“laws”) than a dream or a kaleidoscope?

Not only does the moon not exist when no one is looking at it, it doesn’t exist even when someone *is* looking at it: it’s a hallucination. It is *not* an externally existing object: only conscious beings exist, everything else is a phantom of their imagination. Berkeley’s proof of the existence of God as a being who perceives the moon when no one else is looking is specious. Berkeley didn’t understand the radical implications of his own philosophy.

My own proof of the existence of God runs something like this: if experience makes sense (if the materialistic model is true), then there is no room for any God: there is nothing for Him to do. But no sense *can* be made of experience, therefore, all bets are off: perhaps the stories of Jesus Christ and the Resurrection are true precisely because they are senseless. Why should Christianity be the one true religion of all the religions that have ever been invented and despite the apparent intractability of the Problem of Evil? Perhaps it is as Walker Percy [12] says: we should hold religious doctrine to the same standards of truth that we hold newspaper articles to and not fundamental scientific theories. (According to fundamental science, for example, Joe Biden doesn’t exist as a consistent individual: he is completely different physically and psychologically than he was at the age of five—he is not the same person, he is a process, like a tornado, not a thing. Yet when we read in the papers that he signed such and such a measure into law, we know what it means.)

As opposed to the hard problem of consciousness, the hard problem of matter is solvable, being a “*how* does it work?” question, subject to the scientific method and falsification.

I predict that the next chapter in the history of science will be the study of the hard problem of matter.

Another aspect of the hard problem of matter is the question of why is consciousness so highly correlated with electrochemical activity in the brain.

It is apparently some sort of feedback mechanism like pointing a television camera at its own monitor, resulting in a fractal or infinite regression.

An optimist might even conclude that it would be possible to answer the age-old question, “Why is there something rather than nothing?” which is also an infinite regression (“What created the universe of experience?”...“God,”...“What created God?”...etc.). The question of why there is something rather than nothing is intimately bound up with the question of the existence of consciousness, because if there is no conscious awareness of something, then in a very real sense nothing happened at all: a tree falls in the forest, and no one is around and there is no conscious awareness of it; what happened? Our understanding of what happened is an anthropomorphic metaphor based on the conscious muscular sensations involved in falling down.

By the way, the question of why there is something rather than nothing is the question of why is there any experience of anything at all, not merely why is there a material universe, as writers such as Weatherall [13] believe. (Weatherall’s answer that it has something to do with the quantum vacuum and there is no “nothing” in nature is vacuous: the “quantum vacuum” is a misnomer: it is simply the lowest permissible energy state of matter. In quantum theory, there is no vacuum—nothingness, empty space—but what the Medievals would have called a plenum.)

An aspect of the question of why consciousness is so highly correlated with electrochemical brain activity is the fact that different kinds of conscious organisms experience the world differently: their *worlds* are different, as Nagel [14] has discussed. A bat, for instance, has the sense of echolocation—which humans do not have—and its arms are wings, both of which must color its experience of nature.

Acknowledgement.

My thanks to my friend, Mark Harder, for stubbornly insisting that negative probabilities are not problematic for many years when I just could not “see” it.

References

1. Dirac, P. A. M., “Note on Exchange Phenomena in the Thomas Atom,” Mathematical Proceedings of the Cambridge Philosophical Society, Cambridge University Press, **26** (3) 376-385 (1930)
2. Heisenberg, W., *Physikalische Zeitung* **32** 737-740 (1931)
3. Wigner, E. P., *Physical Review* **40** (5) 749-759 (1932)
4. von Neumann, J., *Mathematical Foundations of Quantum Mechanics*, Princeton University Press, Princeton (1955)
5. Wigner, E. P., “Remarks on the Mind-Body Question,” in Good, I. J. (ed.) *The Scientist Speculates: An Anthology of Partly Baked Ideas*, London: Heinemann (1965)
6. Schrödinger, E., *Naturwissenschaften*, **23** (48) 807-812 (1935)
7. Einstein, A., Podolsky, B., Rosen, N., *Physical Review* **47** (10) 777-780 (1935)

8. Bell, J. S., *Physics* **1** (3) 195-200 (1964)
9. Schommers, W. (ed.) *Quantum Theory and Pictures of Reality* Springer Verlag Berlin chapter 2 (1989)
10. Bell, J. S., "Bertlmann's Socks and the Nature of Reality," (1981) <https://cds.cern.ch/record/281814>
11. Dennett, D. C., *From Bacteria to Bach and Back*, W. W. Norton and Company, New York, London (2017)
12. Percy, Walker, *The Message in the Bottle*, The Noonday Press, Farrar, Strauss and Giroux, New York (1954)
13. Weatherall, J. O., *Void*, University Press, New Haven and London (2016)
14. Nagel, T. *Mortal Questions*, "What is it like to be a bat?" Cambridge University Press (1979)

Appendix

Calculation of the Two-particle Spin Wigner Probability Distribution for the Bell Polarization Experiment.

The spin operator is $\mathbf{s} = \frac{\hbar}{2}\boldsymbol{\sigma}$, where $\boldsymbol{\sigma}$ are the Pauli spin matrices,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The eigenstates are spin up, $|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, and spin down, $|\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

The solution to the eigenvalue problem, $\boldsymbol{\alpha}\mathbf{s}|K\rangle = K|K\rangle$ are,

$$K^\pm = \pm \frac{\hbar}{2} \sqrt{\alpha_1^2 + \alpha_2^2 + \alpha_3^2} := \pm \frac{\hbar}{2} \alpha,$$

$$|K^\pm\rangle = |\uparrow\rangle\langle\uparrow|K^\pm\rangle + |\downarrow\rangle\langle\downarrow|K^\pm\rangle,$$

where,

$$\begin{aligned} \langle\uparrow|K^+\rangle &:= \psi_1^+ = 1, \\ \langle\downarrow|K^+\rangle &:= \psi_2^+ = \frac{\alpha_1 + i\alpha_2}{\alpha_3 + \alpha}, \\ \langle\uparrow|K^-\rangle &:= \psi_1^- = 1, \\ \langle\downarrow|K^-\rangle &:= \psi_2^- = \frac{\alpha_1 + i\alpha_2}{\alpha_3 - \alpha}. \end{aligned}$$

Define,

$$\varphi_1^\pm := \langle K^\pm|\uparrow\rangle, \varphi_2^\pm := \langle K^\pm|\downarrow\rangle,$$

then,

$$\begin{aligned} |\uparrow\rangle &= \varphi_1^+|K^+\rangle + \varphi_1^-|K^-\rangle, \\ |\downarrow\rangle &= \varphi_2^+|K^+\rangle + \varphi_2^-|K^-\rangle, \end{aligned}$$

and $\varphi_m^+\psi_n^+ + \varphi_m^-\psi_n^- = \delta_{mn}$; thus

$$\begin{aligned} \varphi_1^+ &= \frac{\alpha + \alpha_3}{2\alpha}, \varphi_1^- = \frac{\alpha - \alpha_3}{2\alpha}, \\ \varphi_2^+ &= \frac{\alpha_1 + i\alpha_2}{2\alpha}, \varphi_2^- = \frac{-\alpha_1 + i\alpha_2}{2\alpha}. \end{aligned}$$

Define,

$$F_1 = \langle\uparrow|e^{-ias}|\uparrow\rangle, F_2 = \langle\downarrow|e^{-ias}|\uparrow\rangle, F_3 = \langle\uparrow|e^{-ias}|\downarrow\rangle, F_4 = \langle\downarrow|e^{-ias}|\downarrow\rangle.$$

Then,

$$\begin{aligned} F_1 &= \varphi_1^+\psi_1^+e^{-i\alpha\hbar/2} + \varphi_1^-\psi_1^-e^{i\alpha\hbar/2}, \\ F_2 &= \varphi_1^+\psi_2^+e^{-i\alpha\hbar/2} + \varphi_1^-\psi_2^-e^{i\alpha\hbar/2}, \\ F_3 &= \varphi_2^+\psi_1^+e^{-i\alpha\hbar/2} + \varphi_2^-\psi_1^-e^{i\alpha\hbar/2}, \\ F_4 &= \varphi_2^+\psi_2^+e^{-i\alpha\hbar/2} + \varphi_2^-\psi_2^-e^{i\alpha\hbar/2}. \end{aligned}$$

A little calculation shows,

$$\begin{aligned} F_1 &= \cos \frac{\alpha \hbar}{2} - \frac{i\alpha_3}{\alpha} \sin \frac{\alpha \hbar}{2}, \\ F_2 &= \frac{\alpha_1 + i\alpha_2}{\alpha} \cos \frac{\alpha \hbar}{2}, \\ F_3 &= \frac{-i(\alpha_1 - i\alpha_2)}{\alpha} \sin \frac{\alpha \hbar}{2}, \\ F_4 &= \cos \frac{\alpha \hbar}{2} + \frac{i\alpha_3}{\alpha} \sin \frac{\alpha \hbar}{2}. \end{aligned}$$

We see, after some further manipulation that,

$$\begin{aligned} & \frac{1}{2} \sum_{n=1}^4 F_n(\alpha) F_n(\beta) \\ &= \frac{1}{2} \frac{\alpha\beta + \alpha_3\beta_3 + \alpha_1\beta_1 - \alpha_2\beta_2}{\alpha\beta} \cos \frac{(\alpha + \beta)\hbar}{2} \\ &+ \frac{1}{2} \frac{\alpha\beta - \alpha_3\beta_3 + i(\alpha_1\beta_2 + \alpha_2\beta_1)}{\alpha\beta} \cos \frac{(\alpha - \beta)\hbar}{2}. \end{aligned}$$

Note that this cannot be factored in the form, $f(\alpha)f(\beta)$.

For the entangled spin state $|\psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle|\uparrow\rangle + |\downarrow\rangle|\downarrow\rangle)$, the expectation value of $e^{-i(\alpha s_1 + \beta s_2)}$ is $\frac{1}{2} \sum_{n=1}^4 F_n(\alpha) F_n(\beta)$; therefore,

$$\rho(s_1, s_2) = \iiint \frac{1}{2} \sum_{n=1}^4 F_n(\alpha) F_n(\beta) e^{i(\alpha s_1 + \beta s_2)\hbar/2} d^3\alpha d^3\beta,$$

which cannot be resolved into the form $\rho'(s_1)\rho'(s_2)$, where,

$$\rho'(s) = \iiint f(\alpha) e^{i\alpha s\hbar/2} d^3\alpha.$$

■