

Verification of the Theoretical Basis for Determining the Curvature of the Track Axis Using the Moving Chord Method

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ABSTRACT

The paper indicates the possibility of a new approach to the problem of identifying the measured railway route. The currently used methods of identifying the track axis have been characterized, which consist in generating an optimized geometrical layout in the tested area by minimizing the occurring deviations of this layout from the measurement points. The proposed new approach uses the obtained measurement data to determine the existing curvature of the geometric layout. This paper presents a short review of the current activities related to the determination of the curvature of model geometrical layouts and the operated railway track using a new method of determining the curvature, known as the moving chord method. The demonstrated high efficiency of this method should not, however, obscure the fact that the assumptions adopted in it apply – generally speaking – only to a certain, limited range of geometric parameters. The presented analysis showed that the adopted assumptions could be applied to the railway track, because it uses large radii of circular arcs (of the order of several hundred or several thousand meters) and transition curves with a very mild change in curvature. However, the condition must be met that in the calculation procedure appropriate lengths of the moving chord will be used, increasing as the radius of the circular arc increases. Using the test geometric layout, it was clearly demonstrated that the discretization of data used in the moving chord method does not affect the accuracy of the obtained curvature values at individual measurement points.

Keywords: Railway track, Curvature of the track axis, The moving chord method, Verification of theoretical foundations.

INTRODUCTION

During many years of operation, the shape of the existing railway lines is gradually changing and begins to deviate from its originally planned geometric layout. Deviations of the track axis in the horizontal plane may turn out to be particularly unfavorable in terms of passenger comfort and operational safety. In order to eliminate them, it is necessary to determine the current geometric shape of the railway route. This is done on the basis of the designated Cartesian coordinates of the track axis points in the appropriate national spatial reference system. In Poland, for plane coordinates, the PL-2000 system [1] is in force, created on the basis of a mathematically unambiguous assignment of points on the GRS 80 reference ellipsoid [2] to appropriate points on the plane according to the Gauss-Krüger mapping theory [3].

The currently used measurement methods are similar in different railway administrations [4–12]. In classical geodetic techniques, distances and angles are measured using total stations in relation to the spatial geodetic network. Further possibilities are provided by stationary satellite measurements based on the Global Navigation Satellite System (GNSS) technique. This solution does not require the use of a network of the railway ground network points – the measurement systems use the so-called active geodetic networks (e.g., networks of reference stations operating in Real Time Network (RTK) mode [13–15]). Mobile satellite measurement methods are also being introduced, in which, apart from GNSS receivers, Inertial Navigation System (INS) devices [16] are used as supporting devices, and vision methods such as Terrestrial Laser Scanning (TLS) [17]. Research is underway on the possibility of using systems consisting of satellite receivers mounted on various types of vehicles [19–23].

Determining the coordinates of the track axis makes it possible to visualize a given railway route, giving a general orientation about its location. However, since the purpose of the measurements is to determine the geometric parameters (i.e., identification) of the measured route, appropriate calculation algorithms should be used. In the case under consideration (i.e., considering the horizontal plane), the analysis is based on the determined values of the plane eastern coordinates Y_i and the northern X_i coordinates of a given measurement point in the PL-2000 system. However, the proper identification of the track axis is provided by the appropriate graphs referring to the length parameter L . Therefore, in order to create the possibility of further analysis, it is necessary to go to the linear system, i.e. on the basis of Y_i and X_i coordinates, determine the distance (L_i variable) of individual measurement points from the selected starting point $O (Y_0, X_0)$ (i.e. point i_0).

The approach used so far to the problem of identifying the measured route is about generating an optimized geometrical layout in the studied area by minimizing the deviations of this layout from the measurement points, while meeting the appropriate maintenance and operational requirements. The traditional solution to a design problem is largely based on human experience and requires overcoming many difficulties [14,24–28]. To achieve this, many works attempt to automate the discussed process. There are two variants of the procedure: geometric identification and alignment optimization. These variants may be implemented separately or simultaneously.

In terms of geometric identification, an elementary approach is possible, such as the use of discrete approximation methods using splines [29–31] to determine the geometric extent of the various system components. There are also a whole range of other methods that have been presented in works [32–39].

When it comes to alignment optimization, the research methods used vary greatly. Many of them concern transition curves. In paper [40], railway horizontal curves were fitted using the creeping Nelder-Mead simplex method. In [41] horizontal railway curves were reproduced using the Lagrange multiplier method. Papers [42,43] present a general method of curve fitting based on the theory of maximum likelihood estimation and fitting railway transition curves using the Levenberg-Marquardt non-linear optimization algorithm. In paper [44], particle swarm optimization was proposed in order to reconstruct existing railway lines. Other applied alignment optimization methods are presented in [45–51].

In the works mentioned above, geometric identification and alignment optimization were studied separately. In fact (in engineering practice) both of these variants are interrelated. This means that the geometric identification results are used as input for automatic alignment optimization, which in turn – doing the opposite – also changes the identification results. Therefore, in works [13,36,52,53], several methods of combining these variants in an iterative process of track axis reconstruction were proposed. Most of these methods first identify and reconstruct straight line segments (as tangents to curves) and then fit curved line segments.

The possibility of a complete departure from the presented methods of fitting the hypothetical model layout to the points of the track axis, determined by direct measurements, makes it possible to use the obtained measurement data to determine the existing curvature of the geometric layout. For this purpose, an appropriate method for determining the curvature in the railway track had to be developed. Papers [54–56] present relevant analyzes relating to the proposed new method of determining the curvature of the track axis, referred to as the "moving chord method". They concerned the application of this method for model geometric layouts (described with mathematical equations). Papers [57–60] address the issue of its use for the estimation of the horizontal curvature of the axis of the operated railway track on the basis of Cartesian coordinates obtained by direct measurements.

The demonstrated high efficiency of the moving chord method in determining the curvature of the railway track should not obscure the fact that it is based on certain assumptions which, in general, apply only to a certain, limited range of geometrical parameters. This paper tries to clarify these issues by verifying the theoretical basis of the proposed method for determining the curvature of the track axis.

DETERMINATION OF THE CURVATURE OF THE RAILWAY TRACK AXIS BY THE MOVING CHORD METHOD

General Assumptions

From the definition of curvature, it is necessary to operate with the angles of inclination of the tangent to the geometric layout. The measure of curvature κ of road route is the ratio of the angle $\Delta\theta$, by which the direction of the longitudinal axis of the vehicle changes after passing a certain arc, to the length of this arc Δl (Fig. 1). The curvature of the curve K at point M is the limit approached by the ratio of the acute angle $\Delta\theta$ included between the tangents to the curve K at points M and M_1 to the length of the arc Δl , when point M_1 follows the curve K to point M .

$$\kappa = \lim_{\Delta l \rightarrow 0} \left| \frac{\Delta\theta}{\Delta l} \right| = \frac{d\theta}{dl} \quad (1)$$

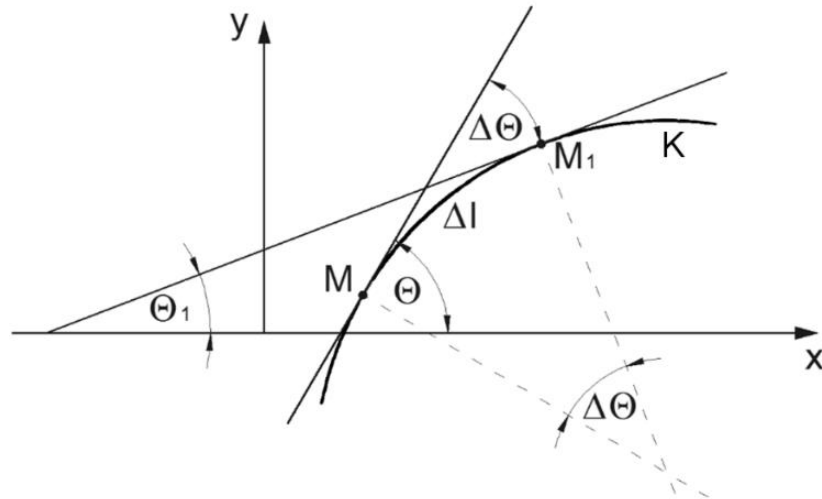


Figure 1. Schematic diagram to explain the concept of curvature of the road route.

If you have an analytical record of a given curve, determining the function $\theta(l)$ and applying formula (1) is not a major problem. However, in a real, operated railway track, there is no such possibility. Therefore, in practice – assuming small values of Δl – a simplified formula can be used

$$\kappa = \frac{\Delta\theta}{\Delta l} \quad (2)$$

The method of determining the angles of inclination of the tangents (and consequently the value of $\Delta\theta$) has not yet been fully resolved.

Due to the difficulties in determining the difference of angles $\Delta\theta$, a concept emerged that when determining the curvature of the track, not tangential but corresponding chords should be used. It should be noted that the use of chords of a certain length, derived from a given point forwards and backwards, has already taken place, among others in [61-63]. However, this was not used to determine the value of the curvature, but to construct a geometric shape consisting of several curves (with clear demarcation points) using the CAGD (Computer Aided Geometric Design) software. In the work [54], a new method of determining the curvature was presented for the first time, consisting in the use of the difference in the inclination angles of a chord of a fixed length occurring at a given point (i.e., the moving chord method).

Determination of the Curvature of the Model Geometric Layouts

The moving chord method, which is theoretical in nature, was verified in [54] on selected model geometric layouts consisting of a circular arc and two symmetrically arranged transition curves (of the same type and length). The types of transition curves and the angles of turning of the route were differentiated. Thanks to the use of mathematical notation, the location of the ends of the chord was uniquely determined. The verification carried out fully confirmed the correctness of the proposed method for determining the curvature of the track axis. In all considered cases, the curvature diagrams showed full compliance with the diagrams constituting the basis for obtaining the corresponding geometric solution.

The discussed method has proven itself in many other considered cases. In the work [55] – also including model layouts – the issues of the influence of chord length on the curvature values obtained and the possibility of determining the location of boundary points between individual geometric elements (i.e. the so-called segmentation points) were addressed. The analyzed variants resulted from the type of transition curves used – in the form of a clothoid, Bloss curve and a new curve from [64]. The curvature values were determined for the assumed moving chord lengths $l_c = 5$ m, 10 m and 20 m. It was found that in each of the variants the graphs of the curvature ordinates coincide with each other. This means that for model layouts, the length of the chord in the range $l_c = 5 \div 20$ m does not play a significant role in determining the curvature and does not limit the use of the discussed method. At the same time, the precision of determining the nature of the curvature and the consistency with the theoretical course on the transition curves are noteworthy.

Since – as it has been shown – the determination of the curvature along the track length is not a special problem, it remains to be clarified the location of the connection points of geometric elements – straight sections with transition curves and transition curves with circular arcs. This may prove particularly important from a practical point of view. The analysis showed that in the case of model (theoretical) geometrical layouts, a chord of length $l_c = 20$ m cannot be used to achieve the intended goal. A chord length of $l_c = 5$ m is suitable, and sometimes an even shorter chord, i.e. $l_c = 2$ m, may be even more advantageous. It follows that in the moving chord method, it is possible to determine the location of segmentation points, but the required length of the chord must be adapted to the type of transition curve.

In the work [56], the universality of the discussed method was indicated – it can be used in relation to both the horizontal and vertical planes. The focus was on the computational basis for the angles of inclination of the moving chord. The analysis of inclination angles in the horizontal plane was carried out in the rectangular coordinate system, on the elementary geometrical layout of tracks. Three special cases were considered, differing in the angle of turn of the route. As part of the analysis, the chord length $l_c = 5$ m was assumed.

The curvature values result from the difference of the inclination angles of the chords derived from point i – forward $\Theta_i^{(+)}$ and backward $\Theta_i^{(-)}$. It was found that the values of these angles are greater if the turning angle of the route is greater. The length of the circular arc maintains a fixed distance between the graphs of both angles (i.e. they are parallel to each other). Along the length of the transition curves, the angle difference decreases to zero on straight sections. It follows that for a circular arc in the horizontal plane, the values of the inclination angles of the moving chord depend on the angle of return of the route, while the difference in inclination angles depends only on the radius of the arc.

From the formal point of view, nothing stands in the way of the moving chord method being also used to determine the vertical curvature. The analysis of the angles of inclination of the moving chord in the vertical plane was carried out on a geometric layout consisting of two symmetrically arranged sections with a uniform inclination of 2,5‰ and a circular arc with a radius of 10,000 m. Having the coordinates of individual points determining the ends of the moving chord, the values of the angles of inclination $\Theta_i^{(+)}$ and $\Theta_i^{(-)}$ of both contacting chords

were determined for each point i . It was found that, similarly to the horizontal plane, the circular arc is fully consistent with the model solution – the curvature of the circular arc is a constant value, and in the extreme regions there is a variation of the curvature, which is undoubtedly related to the length of the adopted chord. A radical shortening of the transition zones was obtained for a 2 m long chord. In the considered case, the radius of the vertical arc is the only factor determining the value of the difference in the inclination angles of the moving chord. However, it should be taken into account that in the case of a circular arc in the vertical plane, the values of the angles of inclination of the moving chord are much smaller than in the horizontal plane, which is related to the range of arc radii used.

Determination of the Curvature of the Operated Railway Track

The verification of the moving chord method on a model geometric layouts carried out in works [54–56] was essentially of a cognitive nature. It confirmed the substantive correctness of the discussed method, but the key issue was its use to determine the horizontal curvature of the axis of the operated railway track on the basis of Cartesian coordinates obtained by direct measurements. From this point of view, the moving chord method ideally corresponds to geodetic measurements carried out in the railway track [58]. In these measurements, we consider a geometric layout with unknown characteristics and we have no possibility to operate with mathematical notation. Therefore, the basic problem here will be to determine the coordinates of the ends of both chords by interpolation carried out in the appropriate intervals. However, as it turns out, it is possible to carry out the entire procedure without resorting to numerical methods.

Paper [57] presents the application possibilities of the moving chord method. It has been shown that based on the estimation of the curvature, it becomes possible to identify the geometrical layouts of the track in the horizontal plane. Measurement data from two geometric track layouts were used, with clearly differentiated values of the radius of the horizontal curve and the nature of the curvature. In both cases, the curvature diagrams obtained are less regular than in the case of model layouts. However, this does not prevent the basic geometrical parameters of the measured layouts from being estimated on their basis. The presented adaptation of the moving chord method to the adopted measurement procedure and the way of practical use of the obtained curvature diagram provide the discussed method with appropriate implementation grounds. The application of the presented procedure should significantly improve the process of identifying the geometric layouts of the track.

In order to verify the moving chord method in the operated railway track, measurements of the Cartesian coordinates of the track axis were carried out using a satellite antenna in a static manner. The distances between the measurement points were about 5 m. Using the appropriate calculation procedure (after assuming the chord length $l_c = 20$ m), the horizontal curvature κ_i of the considered geometric layouts was determined (Fig. 2).

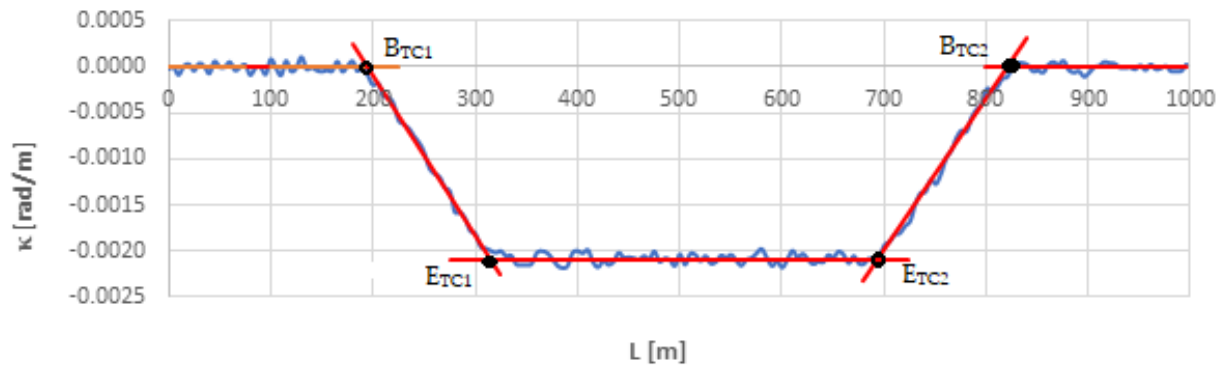


Figure 2. An example of a $\kappa(L)$ graph obtained by the moving chord method.

It was found that the obtained $\kappa(L)$ graphs consist of two types of elements:

- segments oscillating around a horizontal course, which describe the curvature of a fixed value (zero on straight track sections and non-zero on circular curves), and
- segments oscillating around a linear course (i.e. straight lines inclined to the L axis), which describe the variable curvature present in the transition curves.

On the basis of these graphs, it is possible to clearly determine the value of the radius of the circular arc and the length of the transition curves, as well as the location of segmentation points (lying at the connections of straight sections with transition curves and transition curves with a circular arc).

It is assumed that the curvature of the track on straight sections of the geometric layout (left and right in Figure 2) is equal to zero, and the disturbances on the curvature diagram appearing there are the result of the existing deformations and the measurement error. The arithmetic mean κ_{CA} is determined from the selected range of curvature values κ_i indisputably belonging to the circular arc (in the middle region in Figure 2); its inverse determines the value of the radius R of the arc:

$$R = \frac{1}{\kappa_{CA}} \quad (3)$$

There are transition curves on both sides of the circular arc. At the end of the straight line located on the left side of the geometrical layout, there is the beginning of the transition curve $TC1$ (point B_{TC1}), and at the beginning of the straight line on the right side – the beginning of the transition curve $TC2$ (point B_{TC2}). The ends of both curves, i.e. the points E_{TC1} and E_{TC2} , mark the beginning and end of the circular arc, respectively. In order to determine the linear coordinates of the mentioned characteristic points, it is necessary to determine the lines of least squares, describing the regions of the $\kappa(L)$ graph with variable curvature values. These coordinates are determined by the intersection points of the least square lines with the zero curvature (for $L_{B/TC1}$ and $L_{B/TC2}$) and the curvature κ_{CA} (for $L_{E/TC1}$ and $L_{E/TC2}$). The values of the determined $L_{B/TC1}$ and $L_{E/TC1}$ coordinates directly result in the length of the transition curve connected to the straight line located on the left side of the layout.

$$l_{TC1} = L_{E/TC1} - L_{B/TC1} \quad (4)$$

In the same way, the length of the curve connected to the straight line on the right is determined.

The issue of determining the application possibilities of the moving chord method was also addressed in [58]. This paper presents a detailed procedure for determining the curvature of the track axis using the moving chord method, with a view to its practical application to identify the geometrical parameters of the operated railway track. The method of determining the coordinates of the end of the virtual chord derived forwards and backwards, and then determining the curvature occurring at a given measurement point, was explained. The presented procedure consists in the use of given calculation formulas and is sequential in nature. There is no need to use numerical methods. As part of the curvature determination procedure, you can also specify the direction angle values of the route in the appropriate rectangular coordinate system.

In paper [59], the still unexplained issue of selecting the length of the chord, which will be most beneficial when determining the horizontal curvature of the railway track using the moving chord method, was developed. In the railway track – with the wrong selection of the chord length – the occurring horizontal deformations of the track and the error in the measurement of coordinates may result in obtaining irregular curvature diagrams that will be difficult to interpret. In the paper, three test geometrical layouts adapted to the speed of 80 km/h, 120 km/h and 160 km/h were analyzed (the radii of the circular arcs determined as a result of the curvature estimation were approximately 410 m, 880 m and 1,480 m, respectively). The length of the moving chord in the range of 10 ÷ 50 m was considered. Based on the analyzes carried out, it was clearly demonstrated that the length of the chord, used to determine the curvature in the railway track, should depend on the value of the radius of the circular arc. Approximate lengths l_c have been proposed, depending on the R_{CA} radius range. The adjustment of the moving chord method to the adopted measurement procedure and the method of using the obtained curvature diagram, presented in the work, create appropriate application grounds.

Paper [60] presents a detailed algorithm for determining the curvature of the track axis using the moving chord method, and then the procedure for identifying the geometric layout of the operated railway route on the basis of the determined curvature is discussed. The activities carried out have been illustrated with a calculation example, covering a test section of 5.5 km, with five areas of changing the direction of the route. A sequential procedure leading to the solution of the given problem was shown. On the basis of the curvature diagram, the coordinates of the segmentation points were determined, determining the connections of the existing geometric elements with each other.

VERIFICATION OF THE THEORETICAL FOUNDATIONS OF THE MOVING CHORD METHOD

As it seems, there is only one more issue to be clarified in the moving chord method. To make everything clear, it is necessary to return to the initial state and verify the theoretical basis for determining the curvature of the track axis using the moving chord method. The paper [57] presents the assumptions adopted in the discussed method:

- (a) the tangent lines and the corresponding chords are parallel to each other,

- (b) the points of contact project perpendicularly to the center of the given chord,
- (c) the curvature κ_i at a given point i is given by the following formula:

$$\kappa_i = \frac{\Delta\Theta_i}{l_c} \quad (5)$$

where l_c is the chord length used, and the angle $\Delta\Theta_i$ results from the difference of the inclination angles of adjacent chords converging at point i .

Figure 3 shows a schematic diagram of curvature determination using the proposed method. In this case, it concerns a section of a circular arc. To determine the value of curvature at point i , with known Cartesian coordinates, draw a chord of length l_c from this point pointing first backwards and then forwards.

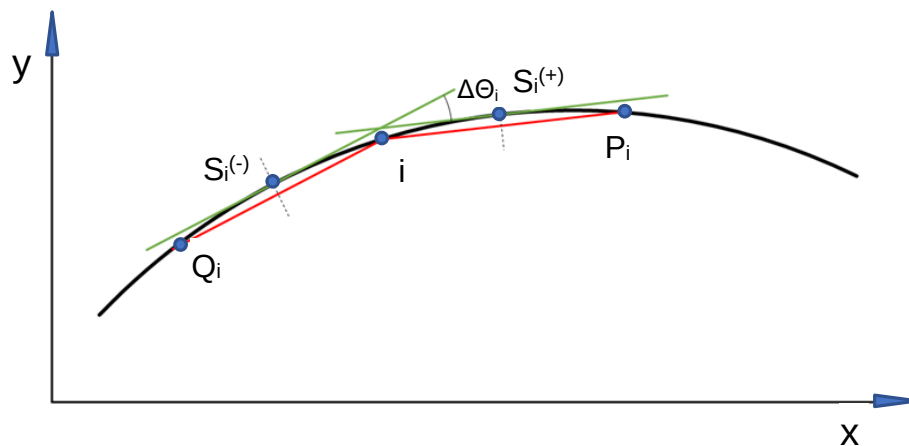


Figure 3. Schematic diagram of curvature determination using the moving chord method.

The end of the backward chord is at the Q_i point. Determining the coordinates of this point makes it possible to determine the angle of inclination of the chord $\Theta_i^{(-)}$. The same angle of inclination has the tangent to the circular arc at the point $S_i^{(-)}$, which is projected perpendicularly to the center of the chord. On the other hand, the end of the forward chord is at P_i point. The coordinates of this point (and the coordinates of point i) determine the angle of inclination of the chord $\Theta_i^{(+)}$, and thus also the angle Θ_i of inclination of the tangent to the circular arc at the point $S_i^{(+)}$, which is projected perpendicularly to the center of the chord. In this situation, the difference in the angles of inclination of the tangent is

$$\Delta\Theta_i = \Theta_i^{(+)} - \Theta_i^{(-)} \quad (6)$$

and the angle of inclination of the tangent at point i (determining the direction angle of the route) is equal

$$\Theta_i = \frac{\Theta_i^{(-)} + \Theta_i^{(+)}}{2} \quad (7)$$

The existing curvature κ_i should be determined from the formula

$$\kappa_i = \frac{\Delta\Theta_i}{S_i^{(-)}S_i^{(+)}} \quad (8)$$

where $S_i^{(-)}S_i^{(+)}$ is the distance between the points $S_i^{(-)}$ and $S_i^{(+)}$ of contact and measured along the arc.

The case shown in Figure 3 includes a segment of a circular arc. As can be seen, the previously given assumptions (a) and (b) are met here. Assumption (c) is not met, because the occurring curvature should be determined not from formula (5), but by formula (8). As can be seen from Figure 3, the distance $S_i^{(-)}S_i^{(+)}$ may turn out to be clearly greater than the corresponding length of the chord l_c .

None of the assumptions made on the transition curve are met. The discrepancy resulting from the non-fulfilment of conditions (a) and (b) in a strict manner is, however, small, while the difference between the lengths $S_i^{(-)}S_i^{(+)}$ and l_c decreases as we move to the initial region of the transition curve.

Therefore, there is no doubt that the assumptions adopted in the moving chord method have a limited range and could only be applied to selected cases. Specifically, this concerned the railway track, where large radii of circular arcs (of the order of several hundred or several thousand meters) and transition curves with a very gentle change of curvature are used.

For a circular arc, the difference Δl_c in length $S_i^{(-)}S_i^{(+)}$ and l_c is given by the following formula:

$$\Delta l_c = S_i^{(-)}S_i^{(+)} - l_c = 2 \cdot R \cdot \arcsin \frac{l_c}{2R} - l_c \quad (9)$$

The graphs of the dependence $\Delta l_c(R)$ for different values of l_c are shown in Figure 4.

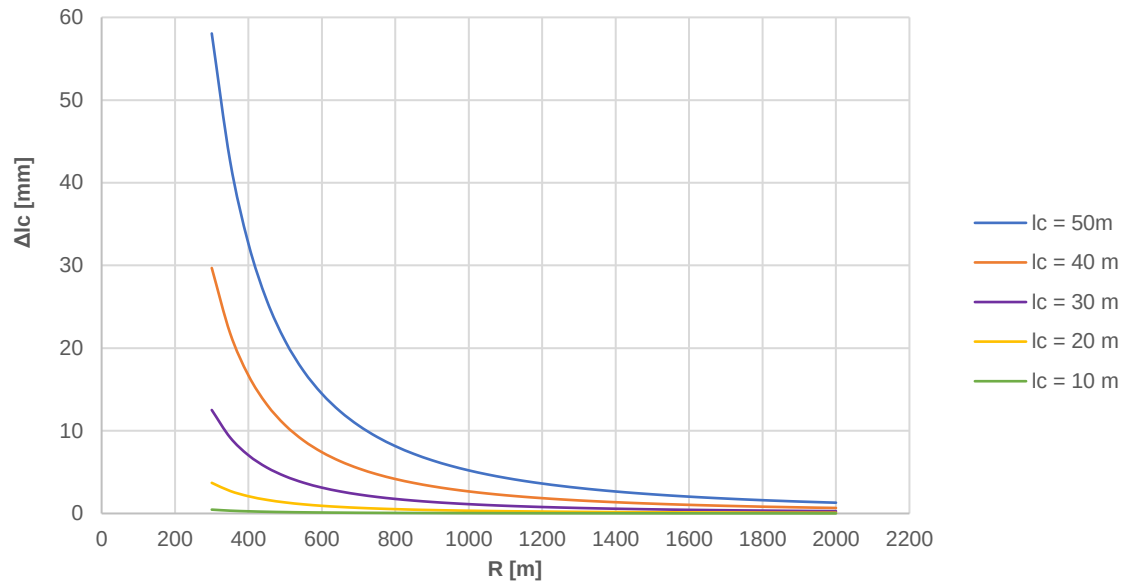


Figure 4. Graphs of $\Delta l_c(R)$ for different l_c values.

As can be seen, the values of Δl_c are the greater if the length of the chord used is greater. However, it is particularly visible for small radii of circular arcs. For $l_c = 10$ m and $l_c = 20$ m, the differences in Δl_c are very small in the whole range of radius R .

At this point, however, it should be noted that in calculation practice, the length of the moving chord cannot be assumed arbitrarily. When choosing the length of the chord, the recommendations formulated in [59] should be followed. Based on the analyzes carried out there, it was clearly demonstrated that the length of the chord, used to determine the curvature in the operated railway track, should depend on the value of the radius of the circular arc. The criterion for selecting the length of the chord was the condition of minimizing deviations of the curvature value from the model course, i.e., zero on straight sections of the track, fixed (as the arithmetic mean) along the length of the circular arc and changing linearly on the transition curve. The following approximate lengths of l_c have been proposed, depending on the range of R values:

- for $R \leq 600$ m $l_c = 20$ m,
- for $600 < R \leq 1000$ m $l_c = 30$ m,
- for $1000 < R \leq 1400$ m $l_c = 40$ m,
- for $R > 1400$ m $l_c = 50$ m.

Assuming the given chord lengths, the graphs of the dependence $\Delta l_c(R)$ presented in Figure 5 are obtained.

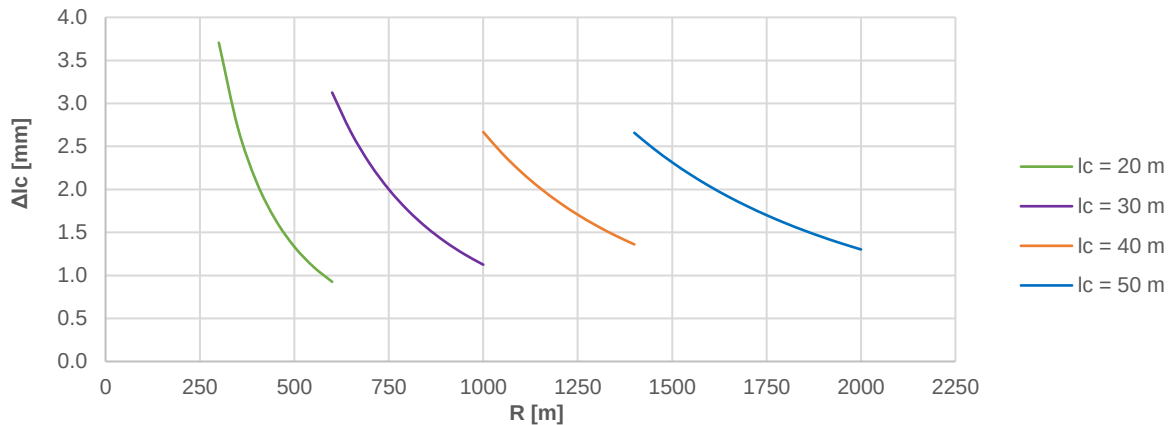


Figure 5. Graphs of dependence $\Delta l_c(R)$ for individual ranges of radius R .

As shown in Figure 5, the adoption of assumptions from work [59] means that in no case the values of Δl_c exceed 4 mm, and in the vast majority of cases – even 3 mm. The curvature values for a circular arc obtained using the formula (5) will therefore be slightly overestimated, and the radius of the arc – slightly underestimated. From a practical point of view, it shouldn't matter much. On the length of the transition curves, the impact of failure to meet the adopted assumptions turns out to be even less significant.

The above considerations largely explain why the cases of determining the curvature using the moving chord method in the operated track, presented in studies [58–60], were fully successful. As you can see, the correctness of the proposed method also has a strong theoretical basis.

THE DESIRABILITY OF CORRECTING THE CURVATURE FORMULA

Regardless of the above statements, it should still be checked whether there is a purpose to correct the curvature formula. Formula (5) is used in the moving chord method, so it would be worthwhile to check what benefits would result from applying formula (8). This formula can be written in the form

$$\kappa_i = \frac{\Delta\Theta_i}{l_c + \Delta l_c} \quad (10)$$

Since it can be assumed that corresponding radius

$$R_i = \frac{1}{\kappa_i} \quad (11)$$

from formulas (9) and (10) it is obtained that

$$\kappa_i = \frac{(\Delta\Theta_i)^2}{2 \cdot l_c \cdot \arcsin \frac{\Delta\Theta_i}{2}} \quad (12)$$

In order to compare the curvature values determined using formulas (5) and (12), it was decided to create a test geometrical layout, designed in mathematical notation in accordance with the principles presented in work [65]. Then, the curvature determination procedure was applied, in which the coordinates of the chord ends were determined precisely, thanks to the knowledge of the $y(x)$ function.

The test geometric layout is shown in Figure 6. It is a symmetrical layout with a length of 1,100 m, a turning angle of the route $\alpha = \pi/3$ rad, adapted to the train speed $V = 120$ km/h. It consists of a circular arc with a radius of 740 m, two transition curves in the form of a clothoid with a length of 120 m and two relatively short (approx. 100 m) straight sections.

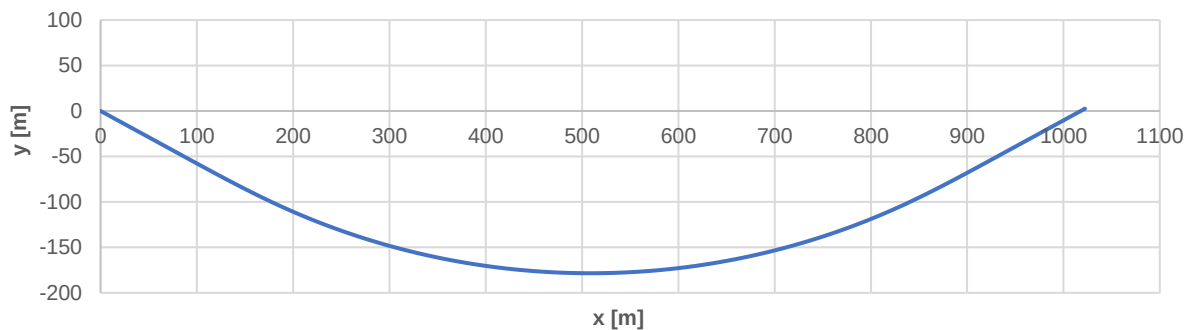


Figure 6. The assumed test geometrical layout.

Identification of the considered layout was carried out on the basis of the curvature determined by the moving chord method using the formula (5); the coordinates of the chord ends were determined using the analytical (i.e., continuous) notation of the geometrical layout. Due to the symmetry, the whole layout was not taken into account, but only its initial fragment of 370 m, containing a straight track section, the first transition curve and part of a circular arc. Calculations were made for the length of the moving chord $l_c = 5$ m, 10 m, 20 m, 30 m, 40 m and 50 m. Since the model (theoretical) geometrical layout was considered, the recommendations given in [60] were not valid, and – in accordance with the findings in [56] – the adoption of the given l_c was justified.

Figure 7 shows the curvature $\kappa(L)$ obtained for the chord $l_c = 5$ m, 30 m, and 50 m. The least squares line describing the curvature on the transition curve, used to determine the location of segmentation points, is marked in red. As can be seen, the curvature diagrams obtained differ only in that as the length of the chord increases, longer and longer non-linear segments appear in the region of the starting and ending points of the transition curve. However, this does not have a major impact on the estimation of the location of these points.

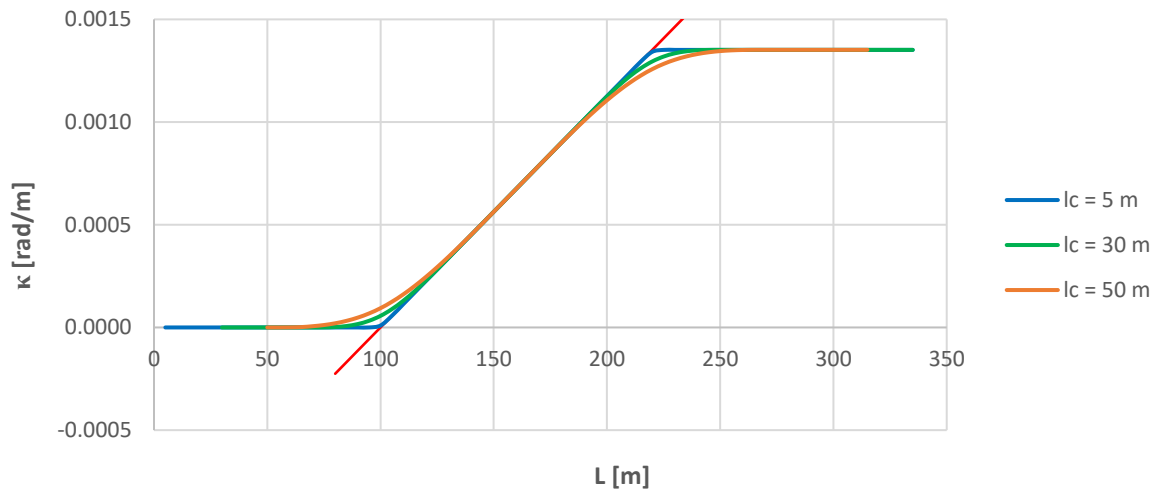


Figure 7. Curvature plot along the length of the considered fragment of the test section for $l_c = 5$ m, 30 m and 50 m (the line of least squares describing the curvature on the transition curve is marked in red).

In the next step of the procedure, the test layout was identified on the basis of the curvature determined using the formula (12), as before, using the analytical record of this layout. From the selected range of curvature values κ_i unquestionably belonging to a circular arc, the arithmetic mean κ_{CA} is determined; its reciprocal gives the value of the radius R .

Table 1 presents a set of numerical values for the curvature on a circular arc (where it is the largest). In addition to the curvature values κ_{CA} and κ_{CA} (and the corresponding radii R and R) obtained for different lengths l_c , absolute differences of the determined curvature $\Delta\kappa = \kappa_{CA} - \kappa_{CA}$ and appropriate percentage differences (relative to curvature κ_{CA}) are also included.

Table 1. Comparison of the values of a circular arc curvature obtained using formulas (5) and (12).

Chord length l_c [m]	Curvature of the arc κ_{CA} [rad/m]	Radius of the arc R [m]	Curvature of the arc κ_{CA} [rad/m]	Radius of the arc R [m]	Curvature difference $\Delta\kappa$ [rad/m]	Percentage difference $\Delta\kappa$ [%]
5	0.0013513539	739.9985925	0.0013513514	740.0000002	0.0000000026	0.0001902
10	0.0013513616	739.9943693	0.0013513514	740.0000001	0.0000000103	0.0007609
20	0.0013513925	739.9774763	0.0013513513	740.0000014	0.0000000411	0.0030440
30	0.0013514439	739.9493184	0.0013513513	740.0000069	0.0000000926	0.0068503
40	0.0013515159	739.9098913	0.0013513513	740.0000219	0.0000001646	0.0121813
50	0.0013516085	739.8591887	0.0013513513	740.0000536	0.0000002573	0.0190394

Table 1 clearly shows that the use of formula (12) leads to greater precision of the calculations, as evidenced by the determined values of the radius of the arc, practically identical to the design radius $R = 740$ m. On the other hand, the differences found between the curvatures κ_{CA} and κ_{CA} are negligible and in the extreme case (for $l_c = 50$ m) they are about 0.02%; in this situation formulas (5) and (8) can be treated as equivalent. It seems that formula (5) can be successfully

used in computational practice to determine the curvature of the track axis. It makes the moving chord method more transparent. From a mathematical point of view, it is not an exact method and is based on certain simplifying assumptions.

IMPACT OF DISCRETIZATION OF MEASUREMENT VALUES

Continuing the consideration of the theoretical aspects of the moving chord method, it was still necessary to address the issue of the impact of operating the coordinates of the track axis set discretely (e.g., every few meters) on the accuracy of curvature determination obtained. This is a typical situation – it applies to all measurements made in a direct way. In order to explain the problem, the test geometric layout described in chapter 4 was used and the effects of identifying this layout were compared based on the curvature determination procedure. First – as the stage I of calculations – the results obtained in chapter 4 were used (where the coordinates of the chord ends were precisely determined), and then (i.e., in the stage II of calculations) the formulas given in works [58,60] were used (for discrete coordinates).

The first stage of calculations carried out made it possible to determine the curvature and radius of the circular arc and to locate the beginning and end of the transition curve (and at the same time the length of the curve) for different values of l_c . Since the actual parameters of the layout were known, it was possible to estimate the accuracy of the method used. As can be seen from the data in Table 2, this accuracy is very high. For each appropriate chord length l_c , the values of curvature κ_{CA} and the radius R of the circular arc, linear coordinates $L_{B/TC}$ and $L_{E/TC}$ – respectively – of the beginning and end of the transition curve, and the length l_{TC} of the transition curve are given.

Table 2. The effects of identifying the test geometrical layout at different chord lengths.

Chord length l_c [m]	Calculation stage	Curvature of arc κ_{CA} [rad/m]	Radius of arc R [m]	Linear coordinate $L_{B/TC}$ [m]	Linear coordinate $L_{E/TC}$ [m]	Length of curve l_{TC} [m]
	Project	0.001351351	740.0000	100.000000	220.000000	120.000000
5	I	0.001351354	739.9986	100.000048	220.000079	120.000031
5	II	0.001351357	739.9972	100.000097	220.000158	120.000061
10	I	0.001351362	739.9944	100.000200	220.000337	120.000137
10	II	0.001351357	739.9972	100.000100	220.000169	120.000069
20	I	0.001351393	739.9775	100.000821	220.001374	120.000553
20	II	0.001351393	739.9775	100.000907	220.001213	120.000307
30	I	0.001351444	739.9493	100.001933	220.003186	120.001253
30	II	0.001351459	739.9409	100.002295	220.003838	120.001543
40	I	0.001351516	739.9099	100.003549	220.005497	120.001948
40	II	0.001351537	739.8986	100.004038	220.006276	120.002238
50	I	0.001351609	739.8592	100.005658	220.007797	120.002139
50	II	0.001351634	739.8451	100.006257	220.008542	120.002285

However, the main subject of the activities undertaken at this point of the work was to explain the effects of operating the coordinates of the track axis set in a discrete way, as is the case when using the results of direct measurements. Therefore, the test layout was discretized by determining its Cartesian coordinates at intervals $\Delta L = 5$ m. Then (as the second stage of

calculations) the discretized layout was identified by the moving chord method, using the procedure described in [58,60].

The obtained graphs of curvature $\kappa(L)$ for different lengths l_c visually do not differ from the graphs obtained in the first stage of calculations. However, of course, there are some numerical differences. A summary of all obtained results is presented in Table 2.

In each case, the curvature obtained by the moving chord method (both in the 1st and 2nd stage of calculations) is greater than the designed curvature. This difference increases as the chord length increases. This, of course, translates into the values of the radius of the circular arc – they are smaller than the radius in the test setup (i.e. $R = 740$ m). However, the differences from the nominal radius are small; for $l_c = 10$ m it is in the order of a few millimeters, then single centimeters, for $l_c = 40$ m – approx. 10 cm, and the largest for $l_c = 50$ m – approx. 15 cm (in this case it is 0.02% radius value). All this confirms once again the very high precision of the discussed method.

The case of the determined linear coordinates of the beginning and end of the transition curve is similar. As l_c increases, the values of L_{BTC} and L_{ETC} also increase, however, the maximum difference from the nominal values (100 m and 220 m, respectively) does not exceed 1 cm. The length of the transition curve resulting from the difference between $L_{B/TC}$ and $L_{E/TC}$ reaches its greatest value for the chord $l_c = 50$ m and is 120.002285 m (deviating from the nominal value $l_{TC} = 120$ m by 0.002%).

Moving on to the issue of the effect of discretization of measurement values (i.e. comparing the results of calculations carried out in stage I and II), Table 2 shows that for $l_c \leq 20$ m the curvature obtained in stage II of calculations is basically the same as the curvature obtained in stage I. For larger l_c , the values of κ_{CA} from stage II are slightly higher. However, the difference is minimal – it is expressed in the values of the radius of the circular arc, which in the extreme case (for $l_c = 50$ m) are less than 15 mm (i.e. 0.002% of the radius value). As for the linear coordinates $L_{B/TC}$ and $L_{E/TC}$ determined in both stages, the differences found are only tenths of a millimeter. This also applies to the l_{TC} values resulting from these coordinates.

In this situation, it can be stated with full responsibility that the discretization of data used in the moving chord method does not affect the final result obtained – the value of curvature occurring at individual measurement points. This was confirmed by an analysis carried out on a model (theoretical) geometrical layout. Therefore, there were no factors that could disturb the inference, i.e. those resulting mainly from the measurement error.

CONCLUSION

The approach used so far to the problem of identifying the measured route is about generating an optimized geometrical layout in the studied area by minimizing the deviations of this layout from the measurement points, while meeting the appropriate maintenance and operational requirements. The traditional solution to a design problem is largely based on human experience and requires overcoming many difficulties. To achieve this, numerous attempts are made to automate the process in question. There are two variants of the procedure: geometric

identification and alignment optimization. These variants may be implemented separately or simultaneously.

An effective solution to the problem of identifying the geometric layout of the track is possible thanks to the use of the obtained measurement data (i.e. Cartesian coordinates) to determine the existing curvature. The high efficiency of the new method of determining the curvature of the railway track axis (the so-called moving chord method), demonstrated in a number of works, should not, however, obscure the fact that it is based on certain assumptions, which – generally speaking – concern only a certain, limited range of geometric parameters. This paper tries to clarify this issue by verifying the theoretical basis of the proposed method of determining the curvature of the track axis.

It was indicated that the adopted assumptions – due to their limited scope – could only be applied to selected cases. Specifically, this concerned the railway track, where large radii of circular arcs (of the order of several hundred or several thousand meters) and transition curves with a very gentle change of curvature are used. The conducted analysis showed the correctness of the adopted assumptions from a practical point of view. However, the condition must be met that in the calculation procedure appropriate lengths of the moving chord will be used, increasing as the radius of the circular arc increases. The discrepancies with the theoretical model are not significant then. On the length of the transition curves, the influence of these discrepancies turns out to be even less significant.

The issue of the influence of operating track axis coordinates set discretely (e.g. every few meters) on the obtained accuracy of curvature determination was also clarified. This is a typical situation that applies to any direct measurement. Using the test geometric layout, it was clearly demonstrated that the discretization of data used in the moving chord method has no effect on the curvature values obtained at individual measurement points.

The activities carried out in the work largely explain why the cases of determining the curvature by the moving chord method in the operated railway track presented in [58–62] were fully successful. As it turns out, the correctness of the proposed method also results from its strong theoretical foundations.

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