

Application of the Displacement Hypothesis to Find the Resistance to A Viscous Fluid Flow Around Bodies

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ABSTRACT

The hypothesis of fluid displacement by a flow around body is investigated, which consists in the fact that the resistance force to a slow flow of a viscous fluid around a body is proportional to the volume of fluid displaced by the body, both due to the replacement of the fluid by the body itself and due to the hydrodynamics of the boundary layer near the body. This hypothesis is confirmed by the example of the flow around a ball and an infinitely thin flat plate. On the basis of this hypothesis, the force of resistance to the across flow around the cylinder and the longitudinal flow around the ellipsoid of rotation are found.

Keywords: displacement hypothesis, Archimedes' law, flow around a body, ball, cylinder, ellipsoid of rotation

INTRODUCTION

Finding the resistance to a viscous fluid flow around bodies is one of the most important problems in hydromechanics. This problem arises when ships move in the aquatic environment, when using oil and gas equipment for pumping oil products, when flying devices move in the atmosphere, when using laboratory equipment in engineering and medicine. In particular, finding resistance to a slow flow of a viscous fluid around bodies is necessary in heat exchangers, when calculating the speed of immersing of various shapes bodies into a liquid, during parachuting, etc.

A slow stationary flow of a viscous fluid around bodies is a rather complex hydrodynamic problem.

Let us consider some regularities of flow around bodies by a continuous medium. We will consider the motion of the body to be sufficiently slow, so that turbulence in the process of flow around the body and separation of the boundary layer with the formation of vortices behind the body can be neglected.

THE EQUATION OF AN IMPULSE

The equation of an impulse in hydrodynamics is a consequence of Newton's second law for a continuous medium. Newton's second law in vector form for a fluid flow can be written as:

$$m\mathbf{a} = \rho W \mathbf{a} = \sum_i \mathbf{F}_i, \quad (1)$$

where $\mathbf{a}$ there is the acceleration of the selected fluid mass m , having the volume W , ρ – is the density of the fluid, $\mathbf{F}_i$ – the forces acting on the selected fluid mass.

Differentiating formula (1) on the fluid volume W , as an independent argument, and assuming the fluid is incompressible, we find:

$$\mathbf{a} = \frac{1}{\rho} \sum_i \frac{\partial \mathbf{F}_i}{\partial W}. \quad (2)$$

In the process of differentiation, it is assumed that all points of the elementary volume dW have the identical acceleration.

Acceleration is the total derivative of speed $\mathbf{V}$ on time t [1], i.e., the sum of the local and convective components:

$$\mathbf{a} = \frac{d\mathbf{V}}{dt} = \frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla) \mathbf{V}. \quad (3)$$

We assume that the main flow of fluid flowing around a stationary body moves in the direction of the X -axis.

Two forces act on the separated mass of fluid: there is driving force $\mathbf{F}_\partial$, and the friction force $\mathbf{F}_f$. Friction force differential $d\mathbf{F}_f$ directed against motion (i.e., velocity $\mathbf{V}_X$) of fluids, so it should be taken with a minus sign. The differential of the driving force is $-d\mathbf{F}_\partial > 0$. This is due to the fact that $\Delta \mathbf{F}_\partial = \mathbf{F}_{2\partial} - \mathbf{F}_{1\partial} < 0$ as when the fluid moves in the positive direction of the X -axis, there is force $\mathbf{F}_{2\partial} < \mathbf{F}_{1\partial}$.

Учитывая знаки дифференциалов сил и (3), уравнение (1) (уравнение импульса) можно представить в виде:

Taking into account the signs of the differentials of forces and (3), the equation (1) (the equation of the impulse) can be represented as:

$$\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla) \mathbf{V} = -\frac{1}{\rho} \frac{d\mathbf{F}_\partial}{dW} - \frac{1}{\rho} \frac{d\mathbf{F}_f}{dW}. \quad (4)$$

Other forces can also act on the fluid volume element dW , for example, gravitational, electric and magnetic, if a metallic fluid moves in an electromagnetic field, etc.

This analysis assumes the absence of such forces.

THE HYPOTHESIS OF DISPLACEMENT

Consider the following hypothesis, which in the future we will call the *hypothesis of displacement*.

The force of resistance to the slow flow of a viscous fluid around a body is proportional to the volume of the fluid displaced by the body, both due to the replacement of the fluid by the body itself and due to the hydrodynamics of the boundary layer near the body.

This hypothesis is a generalization of Archimedes' law for a body immersed in a fluid. But Archimedes' law applies only to a fluid that is steady relative to a body immersed in it.

Let W is the volume of fluid displaced by the body in the flow. We believe that the friction force of a fluid against a body is proportional to the volume of the displaced fluid $F_f \sim W = W_1 + W_2$ (or $dF_f \sim dW = dW_1 + dW_2$), where W_1 is the volume of fluid displaced by the body itself, and W_2 is the volume of fluid displaced due to the hydrodynamics of the boundary layer near the body. Indirectly on the relationship between the friction force and the volume of the displaced fluid is indicated by the fact that if $dW = 0$, then and $dF_f = 0$.

Otherwise in equation (4) $\frac{dF_f}{dW} \rightarrow \infty$, that is impossible.

Let's test the displacement hypothesis for two limiting cases. Firstly, check for the flow of so-called volumetric bodies, where the ratio of the volume of the body to the surface is large. First of all, for the ball, where is $W_1 \gg W_2$. Secondly, for the flow of bodies with a small (in the limit zero) ratio of the volume of the body to its surface, in particular, for a flat infinitely thin plate, where is $W_2 \gg W_1 = 0$.

THE FLOW OF INCOMPRESSIBLE VISCOUS FLUID AROUND THE BALL

The slow stationary flow around a ball with a viscous fluid was first investigated by Stokes in 1851 [2]. The solution of the Stokes problem allows us to obtain a result of exceptional importance – the force of the viscous resistance of the ball when flowing around it with a fluid. This is due to the fact that this result finds application in many fields of science and technology. However, the deduction of the Stokes formula, despite its simplicity, is quite complicated. Therefore, in many monographs it is or not given at all [3], although the formula itself is used, or it is not given in full [4].

We will show how, based on the vector form of Newton's law for a viscous fluid and the displacement hypothesis it is possible to obtain the Stokes formula relatively simply and physically clearly. The vector form of Newton's law for the friction force in a viscous fluid has the form [5]:

$$d\mathbf{F}_f = \eta d\mathbf{S} \times \text{rot}\mathbf{V}, \quad (5)$$

where $d\mathbf{S}$ there is the vector of the contact area of the fluid layers (or the contact area of the fluid and the streamlined body), η is the viscosity coefficient – in this case, a scalar value.

For convenience, we will carry out all transformations in the Cartesian coordinate system. Note that at a potential flow, i.e. at $\text{rot}\mathbf{V} = 0$, the friction force is $d\mathbf{F}_f = 0$, i.e. the flow is isentropic [6]. There is no friction between the fluid layers not only in the case of an ideal liquid but also at an equal velocity of these layers.

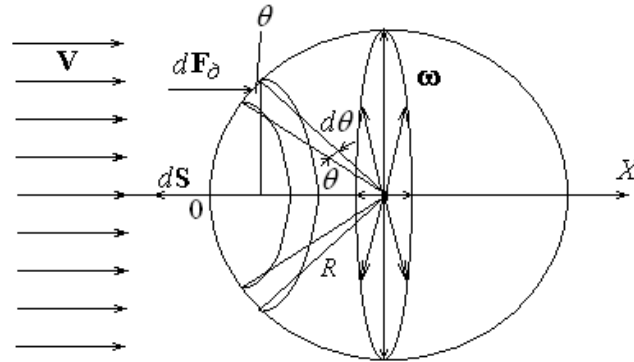


Fig. 1. Flowing around the ball with a viscous fluid

Let us consider on the basis of the impulse equation (4) a slow stationary flow around a ball with a viscous fluid at a velocity $\mathbf{V}$, Fig. 1.

We use the standard approximation for the Stokes problem [7] in which the driving force and the friction force are of primary importance. In this case the equation (4) is transformed to the form:

$$\frac{d\mathbf{F}_\partial}{dW} = -\frac{d\mathbf{F}_f}{dW}. \quad (6)$$

We also select, following [7], on the surface of the ball, Fig. 1, a belt with an area:

$$dS = 2\pi R^2 \sin \theta d\theta, \quad (7)$$

where R there is the radius of the ball, θ is the angle at which the belt on the ball is located.

When a fluid flows around a ball the angular velocity vector of the fluid particles ω , flowing around the ball is always perpendicular to the differential of general vector of the belt area $d\mathbf{S}$, Fig. 1. Therefore, the modulus of the friction force differential (5) can be written as:

$$dF_f = 2\eta\omega dS. \quad (8)$$

When writing (8) it is taken into account that $\text{rot}\mathbf{V} = 2\boldsymbol{\omega}$ [7].

Find the right side of equation (6) in scalar form:

$$-\frac{dF_f}{dW} = -\frac{2\eta\omega dS}{dW} = -6\eta \frac{V}{R^2}. \quad (9)$$

When finding (9) the following are used: the formula $\omega = \frac{V}{R}$, and the formula $dW = \frac{1}{3}RdS$ for the elementary volume of the fluid displaced by the ball. The minus reflects the fact that the friction force F_{mp} is directed against the force F_∂ driving the fluid. Note that in the approximation under consideration $W_1 \gg W_2$, we neglected the boundary layer, so when the ball flows around, only one value of the linear velocity V is used.

The use of the formula for the volume differential $dW = \frac{1}{3}RdS$ is determined by the following.

In equation (6), the volume dW , standing in the denominator of the term determining friction in a fluid, is covered by the surface area dS , used in formula (5). Surface elements that are not flowed by the fluid, and the velocity of the fluid to which is perpendicular, may not be considered. Therefore, to the area of the belt on the ball dS corresponds the volume of the hollow cone dW which the base is the belt dS .

Find the scalar form of the left side of equation (6):

$$\frac{dF_\partial}{dW} = \frac{dF_\partial}{dXdS \cos \theta}. \quad (10)$$

In (10) it is taken into account that the driving fluid force dF_∂ acts in the direction of the coordinate axis X on the area $dS \cos \theta$, Fig. 1.

Equating (9) to (10) and taking into account (7), we find:

$$dF_\partial = -6\eta \frac{V}{R^2} dXdS \cos \theta = -6\pi\eta V \sin 2\theta d\theta dX \quad (11)$$

Integrating (11) by the angle θ and by the X coordinate, we have:

$$F_\partial = -6\pi\eta V \int_0^{\frac{\pi}{2}} \sin 2\theta d\theta \int_0^R dX = -6\pi\eta RV. \quad (12)$$

When substituting the integration limits, it is taken into account that the force F_∂ acts on the left half of the ball, Fig. 1. The origin of the X coordinate is taken on the surface of the ball on the left side.

Since the force of resistance to the movement of the fluid from the ball – the Stokes force F_X – is equal to the force $-F_\partial$, then we can write:

$$F_X = 6\pi\eta RV. \quad (13)$$

Thus, for the ball, the application of the displacement hypothesis leads to the correct result.

SLOW FLOW AROUND A THIN PLATE WITH A VISCOUS FLUID

Let us analyze, using the displacement hypothesis, the slow flow around a flat infinitely thin plate with a viscous fluid, Fig. 2.

Let's find the value $-\frac{dF_f}{dW}$ in equation (4) for a thin flat plate of width b (perpendicular to Fig. 2), streamlined along the X -axis on both sides. As mentioned earlier, in this case, all displacement is carried out due to the hydrodynamics of the flow and $dW_2 \gg dW_1 = 0$ or $dW = dW_2$.

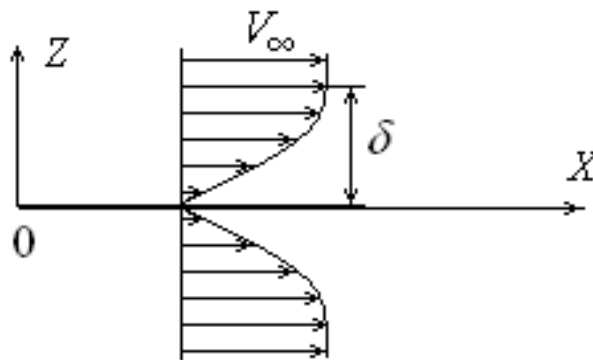


Fig. 2. Flow around the plate with a viscous fluid

The friction force during flow around, taken into account the two sides of the plate, is equal to:

$$dF_f = 2\tau_0 b dX, \quad (14)$$

where τ_0 is the friction stresses on the plate.

Due to the fact that our task is not a complete analysis of the flow near a flat plate, we will use the finished result [4] for the friction stress:

$$\tau_0 = \frac{\rho V_\infty^2}{2} \frac{\delta_2}{X} = \frac{\rho V_\infty^2}{2} \frac{\alpha_1}{\alpha_2} \frac{\delta_1}{X}, \quad (15)$$

where V_∞ is the velocity of the fluid away from the plate, $\delta_1 = \int_0^\infty \left(1 - \frac{V}{V_\infty}\right) dZ$ is the thickness of the displacement of the fluid in the boundary layer (Z – coordinate across the plate), is the thickness of the impulse loss. The values $\alpha_1 = \frac{\delta_2}{\delta}$ and $\alpha_2 = \frac{\delta_1}{\delta}$, where δ is the thickness of the boundary layer, Fig. 2.

Substituting (15) into (14), we find:

$$dF_f = 2 \frac{\rho V_\infty^2}{2} \frac{\alpha_1}{\alpha_2} \frac{\delta_1}{X} b dX = \frac{\rho V_\infty^2}{2} \frac{\alpha_1}{\alpha_2} \frac{dW_2}{X}. \quad (16)$$

For a flat plate the elementary volume of the displaced fluid in the boundary layer is $dW_2 = 2\delta_1 b dX$. The coefficient two takes into account that the displacement of the fluid occur from both sides of the plate.

As $dF_f \sim dW_2$, then in fact the displacement hypothesis for the slow flow around a flat plate is proved. Despite this, we will conduct a further analysis of the flow.

According to (16), taken into account that $dW = dW_2$, for the plate we have:

$$-\frac{dF_f}{dW} = -\frac{\rho V_\infty^2}{2X} \frac{\alpha_1}{\alpha_2} = -\frac{\rho V_\infty^2}{2XH}, \quad (17)$$

where is $H = \frac{\alpha_2}{\alpha_1} = \frac{\delta_1}{\delta_2}$ the form-parameter. For a laminar boundary layer the form-parameter ranges from 2.3 to 3.5. For a turbulent boundary layer it is from 1.3 to 2.2 [4].

Comparing (15) and (17), we find:

$$-\frac{dF_f}{dW} = \frac{\tau_0}{\delta_1}. \quad (18)$$

Let's find the value $-\frac{dF_\partial}{dW}$ in equation (4). If we introduce a pressure gradient $\frac{\partial P}{\partial X}$ and take into account that for the fluid volume element $dW = b dX dZ$ the differential of the driving force is $dF_\partial = b dP dZ$ then $\frac{dF_\partial}{dW} = \frac{\partial P}{\partial X}$. When using approximation (6), we obtain:

$$\rho V_\infty \frac{\partial V_\infty}{\partial X} = \frac{\tau_0}{\delta_1}. \quad (19)$$

There is taken into account the known relation for the flow in the boundary layer $\frac{\partial P}{\partial X} = -\rho V_\infty \frac{\partial V_\infty}{\partial X}$ [4]. The resulting equation (19) is the equation of impulse [4] for the flow around a flat plate:

$$\frac{\tau_0}{\rho} = \frac{d}{dX} (V_\infty^2 \delta_2) + \delta_1 V_\infty \frac{dV_\infty}{dX} \quad (20)$$

provided that the first term in the right part is neglected (slow flow).

Equation (20) can be given the following form [7]:

$$\frac{d\delta_2}{dX} + (2 + H) \frac{\delta_2}{V_\infty} \frac{dV_\infty}{dX} = \frac{\tau_0}{\rho V_\infty^2}, \quad (21)$$

which is called the equation of impulse or the integral Karman relation [8].

Thus, using the displacement hypothesis for slow flow around a flat plate leads to correct results.

FLOW AROUND THE CYLINDER WITH A VISCOUS LIQUID

It is believed that for a long cylinder located across the flow the problem of its slow flow around a viscous fluid cannot be solved even for very small Reynolds numbers at neglecting the nonlinear inertial terms in the equation (4). Taking into account the inertial terms in the equation (4) makes it possible to solve the problem with boundary conditions at infinity but already in the nonlinear approximation [6].

The statement about the impossibility of solving the linear problem of cylinder flow around is called the **Stokes paradox** [10]. It is impossible to satisfy simultaneously the condition of maintaining undisturbed velocity at infinity and the condition of adhesion of liquid particles to the surface of the cylinder.

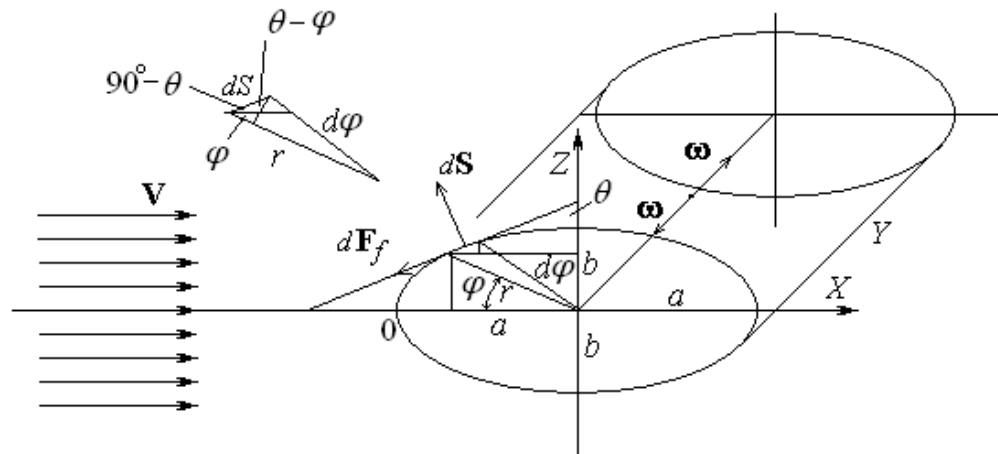


Fig. 3. Flow around the cylinder with a viscous fluid

Let us consider the problem of slow flow around a cylinder with a viscous fluid using the vector form of Newton's law for a viscous fluid in the form (5) and the displacement hypothesis for slow flow around bodies, Fig. 3.

Let a rigid cylinder having an arbitrary profile in cross section with a center of symmetry flow around a viscous fluid.

Let's select an elementary area on the surface of the cylinder:

$$dS = Y \frac{r}{\cos(\theta - \varphi)} d\varphi, \quad (22)$$

where r is the variable radius of the cylinder, φ is the angle at which the area dS is located on the cylinder, θ is the angle determining the position of the tangent to the cylinder section, Y is the length of the cylinder.

When a fluid flows around a cylinder, the vector of the angular velocity of the liquid particles ω flowing around the cylinder is directed along the axis of the cylinder and is always perpendicular to the vector of the area differential dS , Fig. 3. Therefore, the modulus of the friction force is:

$$dF_f = 2\eta\omega dS = 2\eta\omega Y \frac{r}{\cos(\theta - \varphi)} d\varphi. \quad (23)$$

Let's find the value $-\frac{dF_f}{dW}$ in equation (4) in scalar form:

$$-\frac{dF_f}{dW} = -\frac{2\eta\omega Y r d\varphi}{(1/2)Y r^2 \cos(\theta - \varphi) d\varphi} = -4\eta \frac{\omega}{r \cos(\theta - \varphi)} = -4\eta \frac{V}{r^2 \cos(\theta - \varphi)}. \quad (24)$$

When finding (24), formulas for the angular velocity of the flowing around fluid $\omega = \frac{V}{r}$, and the elementary volume of the fluid displaced by the cylinder $dW = Y \frac{1}{2} r^2 d\varphi$ were used. The principle of choosing the elementary volume dW is the same as in the case of the flow around the ball, see point 4.

Let's find the scalar form of the quantity $-\frac{dF_\partial}{dW}$ in equation (4):

$$-\frac{dF_\partial}{dW} = -\frac{dF_\partial}{dX dS \cos \theta}. \quad (25)$$

In (25) it is taken into account that the driving fluid force dF_∂ acts in the direction of the X coordinate axis on the area $dS \cos \theta$.

As for the ball, we use the linear approximation (6) in the equation of impulse (4).

Equating the modulus (25) to (24) and taking into account (22), we find:

$$dF_\partial = -4\eta \frac{V}{r^2 \cos(\theta - \varphi)} dX dS \cos \theta = -4\eta \frac{V}{r \cos^2(\theta - \varphi)} Y \cos \theta d\varphi dX. \quad (26)$$

Integrating (26) by the angle φ and by the X coordinate, we have:

$$F_\partial = -4\eta V Y \int_0^{\frac{\pi}{2}} \int_0^a \frac{\cos \theta dX d\varphi}{r \cos^2(\theta - \varphi)}. \quad (27)$$

where a is half the size of the cylinder profile in the direction of the X axis, Fig. 3.

When setting the integration limits, it is taken into account that the force F_∂ acts on the left half of the cylinder, Fig. 3. The limits of integration by the angle φ are determined that the

equalization of values (24) and (25) is carried out along the upper left quarter of the cylinder area, Fig. 3. The origin of the X coordinate, as with the flow around the ball is taken on the surface of the cylinder on the left side.

In the case of flow around a circular cylinder with a radius $R = a = \text{const}$ the corners $\theta = \varphi$. Therefore:

$$F_{\partial} = -4\eta \frac{V}{R} Y \int_0^{\frac{\pi}{2}} \cos \varphi d\varphi \int_0^R dX = -4\eta VY. \quad (28)$$

Consequently, the force of resistance to the movement of the fluid from the circular cylinder is equal to:

$$F_X = 4\eta VY. \quad (29)$$

The resulting formula (29) is unacceptable from a physical point of view, because at the radius of the cylinder $R = 0$, the resistance force $F_X \neq 0$ [6].

To exclude such an effect researcher most often go to the full equation (4), taking into account the nonlinear terms [6].

What is the inaccuracy of the formulation of the cylinder flow around problem from the point of view of the displacement hypothesis?

A long cylinder, unlike a ball, does not satisfy the requirement $W_1 \gg W_2$. The geometry of the surface of a cylinder and a ball of the same radius differs significantly in the following. The surface of the ball is limited by the size of the radius. The lateral surface of a cylinder of the same radius can be of any size due to its length Y .

Therefore, the displacement of the fluid due to the hydrodynamics of the flow at a large size Y may exceed the displacement due to the volume of the cylinder. Consequently, in order to correctly apply the displacement hypothesis and satisfy the requirement $W_1 \gg W_2$ it is necessary to limit the length of the cylinder, make it commensurate with the across size of the cylinder. As the size of the cylinder a along the flow, Fig. 3, can be preserved for the plane, see paragraph 5, we use the size of the cylinder $2b$ across the flow as the scale of the Y value. Let's assume $Y = 2\xi b$, where ξ is a constant coefficient of a relatively small value (greater than one), but so that the edge effects at the ends of the cylinder during its flow around can be neglected.

In this case, the formula (27) for the force F_{∂} takes the form:

$$F_D = -8\eta V \xi b \int_0^a \int_0^{\frac{\pi}{2}} \frac{\cos \theta dX d\varphi}{r \cos^2(\theta - \varphi)}. \quad (30)$$

The force of resistance to the movement of the fluid from the circular cylinder in this case $a = b = R$ will be equal to:

$$F_X = 8\eta V \xi R. \quad (31)$$

Formula (31) satisfies the condition: at $R = 0$ the value $F_X = 0$ if the displacement thickness due to the hydrodynamics of the flow is neglected.

Let us estimate the accuracy of the obtained result (31). The dependence of the cylinder resistance coefficient on the Reynolds number in the approximation (31) has the form:

$$C_W = \frac{F_X}{\frac{1}{2} \rho V^2 F} = \frac{8}{\text{Re}}, \quad (32)$$

where $\text{Re} = \frac{\rho V d}{\eta}$, and $F = 2\xi R d$ is the frontal area of the cylinder flow around [4], the diameter of the circular cylinder $d = 2R$.

Formula (32) reflects well the experimental curve [4] in the range $\text{Re} = 0.4 \pm 0.2$. It is interesting to compare formula (32) with the C.W. Oseen formula $C_W = \frac{8}{\chi \text{Re}}$ where

$\chi = \frac{1}{\pi} \ln \frac{7.406}{\text{Re}}$ [4], obtained on the basis of solving the nonlinear equation of impulse at the flow around the cylinder. In the considered range of Reynolds numbers the coefficient $\chi = 0.8 \div 1.15$.

To expand the range of Reynolds numbers to $0.1 < \text{Re} < 2$, in the formula (32) there must be value $\text{Re}^{0.8}$ in the denominator. For a flat plate where fluid displacement occurs only due to the hydrodynamics of the flow is $C_W = \frac{1.328}{\text{Re}^{0.5}}$ [4]. Therefore, it can be concluded that the

inaccuracy of formula (32) in comparison with experimental data is associated with the inaccuracy of the condition $W_1 \gg W_2$, i.e. it is determined by a significant displacement of the fluid in the boundary layer during the flow around the cylinder.

FLOW AROUND THE ELLIPSOID OF ROTATION WITH A VISCOUS FLUID

Let us consider, on the basis of the displacement hypothesis, the flow around the ellipsoid of rotation by a viscous incompressible fluid. We assume that the ellipsoid of rotation has a longitudinal section along the flow, shown in Fig. 4. The value a there is the semi-axis of the ellipsoid along the flow, b is the semi-axis of the ellipsoid of rotation perpendicular to the flow. As in the case of the ball, the calculation is based on equation (6).

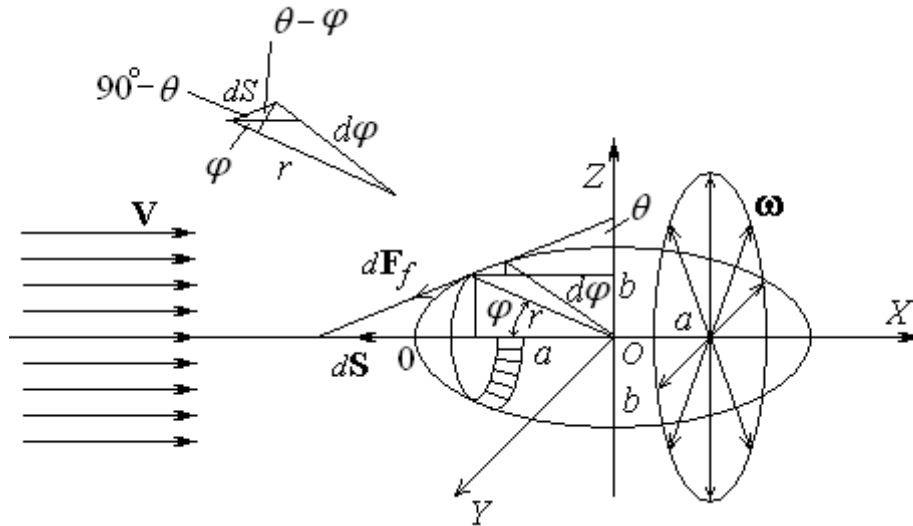


Fig. 4. Flow around the ellipsoid of rotation by a viscous fluid

Let's consider on the surface of the ellipsoid of rotation, Fig. 4, a circular belt with an area of:

$$dS = 2\pi r \sin \varphi \frac{r}{\cos(\theta - \varphi)} d\varphi = \frac{2\pi r^2 \sin \varphi}{\cos(\theta - \varphi)} d\varphi. \quad (33)$$

When a fluid flows around an ellipsoid of rotation, the vector of the angular velocity of the liquid particles ω flowing around the ellipsoid is always perpendicular to the general vector of the differential area of the belt dS , Fig. 4. The vector ω lies in the YOZ plane, Fig. 4 (similar to Fig. 1). Therefore, the modulus of the friction force differential (5) can be written as:

$$dF_f = 2\eta \omega dS. \quad (34)$$

The volume element corresponding to this friction force is equal to:

$$dW = \frac{1}{3} r dS \cos(\theta - \varphi). \quad (35)$$

Taking into account $\omega = \frac{V}{r}$, we find the scalar form of the right side of equation (6):

$$-\frac{dF_f}{dW} = -6\eta \frac{V}{r^2 \cos(\theta - \varphi)}. \quad (36)$$

The scalar form of the left side of equation (6) has the form similar to (10):

$$\frac{dF_\delta}{dW} = \frac{dF_\delta}{dXdS \cos \theta}, \quad (37)$$

Equating (36) and (37), taking into account (33), we find the force acting on the ellipsoid:

$$F_\delta = -12\pi\eta V \int_0^a dX \int_0^{\frac{\pi}{2}} \frac{\cos \theta \sin \varphi d\varphi}{\cos^2(\theta - \varphi)} = -12\pi\eta Va \int_0^{\frac{\pi}{2}} \frac{\cos \theta \sin \varphi d\varphi}{\cos^2(\theta - \varphi)}. \quad (38)$$

The origin of the X coordinate is taken on the surface of the ellipsoid on the left side.

The resistance force can be expressed in terms of the radius of the equivalent ball:

$$F_X = -F_\delta = 6\pi\eta\rho_0 V, \quad (39)$$

where ρ_0 in this case is the equivalent radius:

$$\rho_0 = 2b \int_0^{\frac{\pi}{2}} \varepsilon \frac{\cos \theta \sin \varphi d\varphi}{\cos^2(\theta - \varphi)}. \quad (40)$$

The value $\varepsilon = \frac{a}{b}$ is the relative elongation of the ellipsoid shape.

The relationship between the angles θ and φ is easy to find based on the geometric meaning of the derivative of the ellipse equation $\frac{(a-X)^2}{a^2} + \frac{Z^2}{b^2} = 1$. Taking into account that both angles

are acute, we have $\operatorname{tg} \theta = \frac{a^2}{b^2} \operatorname{tg} \varphi$. Using the relation $\operatorname{tg} \theta = \varepsilon^2 \operatorname{tg} \varphi$, we transform the equivalent radius:

$$\rho_0 = 2b \int_0^{\frac{\pi}{2}} \varepsilon \frac{\cos \theta \sin \varphi d\varphi}{\cos^2(\theta - \varphi)} = 2b \int_0^{\frac{\pi}{2}} \varepsilon \frac{\cos[\operatorname{arctg}(\varepsilon^2 \operatorname{tg} \varphi)] \sin \varphi d\varphi}{\cos^2[\operatorname{arctg}(\varepsilon^2 \operatorname{tg} \varphi) - \varphi]}. \quad (41)$$

If $a = b$, that is $\varepsilon = 1$, the angles is $\theta = \varphi$. From the formula (40) it follows that in this case is

$$\rho_0 = 2a \int_0^{\frac{\pi}{2}} \cos \varphi \sin \varphi d\varphi = a. \text{ Therefore, for the resistance (39) we obtain the Stokes formula (13) for the flow around the ball of viscous fluid.}$$

Integral (41) at $\varepsilon \geq 1$, i.e. for an elongated ellipsoid of revolution is integrated in quadratures. Originally we get rid of the arctangent using the formula $\cos[\operatorname{arctg}(\varepsilon^2 \operatorname{tg} \varphi)] = \frac{1}{\sqrt{1 + \varepsilon^4 \operatorname{tg}^2 \varphi}}$.

After simple transformations, the integral (41) gets converted to:

$$\rho_0 = 2b \int_0^{\frac{\pi}{2}} \varepsilon \frac{\sqrt{1 + \varepsilon^4 \operatorname{tg}^2 \varphi} \sin \varphi d\varphi}{\cos^2 \varphi (1 + \varepsilon^2 \operatorname{tg}^2 \varphi)^2}. \quad (42)$$

Next by substitution $Z = \cos^2 \varphi$ we move on to algebraic functions:

$$\rho_0 = b \int_0^1 \varepsilon \frac{\sqrt{Z + \varepsilon^4(1 - Z)} dZ}{[Z + \varepsilon^2(1 - Z)]^2}. \quad (43)$$

Then, using the substitution $t^2 = Z + \varepsilon^4(1 - Z)$, we go to the rational algebraic functions and find:

$$\rho_0 = b \frac{\varepsilon^2 + 1}{\varepsilon^2 - 1} \operatorname{arctg} \left(\frac{\varepsilon^2 - 1}{2\varepsilon} \right). \quad (44)$$

At $\varepsilon \rightarrow 1$ the equivalent radius, as in (44) is equal $\rho_0 = b$. At $\varepsilon \rightarrow \infty$ the equivalent radius is equal $\rho_0 \rightarrow \frac{\pi}{2} b$. Using (39) we find the resistance force for very elongated ellipsoid of revolution when it flowing around of a viscous incompressible fluid along a major axis:

$$F_X = 3\pi^2 \eta b V. \quad (45)$$

In the calculation according to the formula (45) the force resistance is low. This is due to the fact that as the ellipsoid lengthens the role of the boundary layer in fluid displacement increases.

The dependence of the resistance coefficient of the ellipsoid in the approximation (45) on the Reynolds number has the form:

$$C_W = \frac{F_X}{\frac{1}{2} \rho V^2 \pi b^2} = \frac{12\pi}{\text{Re}}, \quad (46)$$

where $\text{Re} = \frac{\rho V d}{\eta}$, the diameter of the maximum cross-section of the ellipsoid of rotation $d = 2b$, ρ is the density of the liquid.

In Fig. 5 shows the graph of calculation by formula (44) of the relative equivalent radius $\frac{\rho_0}{b}$ of the ellipsoid of rotation (curve 1) depending on the relative elongation $\varepsilon = \frac{a}{b}$ of the shape of this ellipsoid.

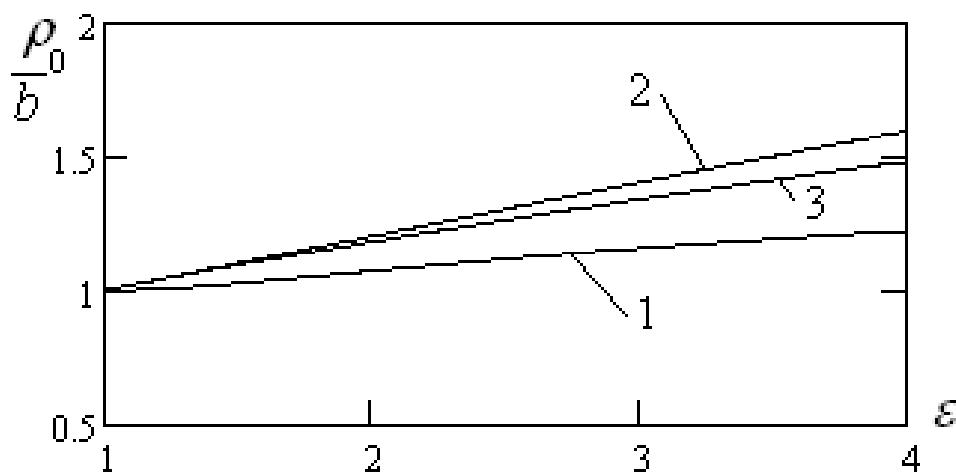


Fig. 5. Dependences of the relative equivalent radius of the ellipsoid of rotation on the relative elongation of its shape

For comparison, the results of the calculation according to the authors' formula [11]

$$\frac{\rho_0}{b} = \frac{8}{3} \left[-\frac{2\varepsilon}{\varepsilon^2 - 1} + \frac{2\varepsilon^2 - 1}{(\varepsilon^2 - 1)^{\frac{3}{2}}} \ln \left(\frac{\varepsilon + \sqrt{\varepsilon^2 - 1}}{\varepsilon - \sqrt{\varepsilon^2 - 1}} \right) \right]^{-1} \quad (\text{curve 2}) \text{ and [12] according to the}$$

formula $\frac{\rho_0}{b} = \sqrt{\frac{1}{5}} (3 + 2\varepsilon)$ (curve 3) are also shown.

Curve 1 gives lower results than curves 2 and 3. Apparently, this is due to the increasing influence of fluid displacement in the boundary layer as the ellipsoid lengthens, but the final choice between the calculations shown in Fig. 5 can only be made as a result of experiments.

CONCLUSION

Thus, the use of the displacement hypothesis makes it possible to estimate the resistance of bodies of various shapes to slow flow around them with a viscous fluid. Due to the fact that the resistance of the body is determined by the displacement of the fluid, both by the body itself and by the boundary layer on the streamlined body, it becomes possible to differentially determine the parameters of the boundary layer, for example, the displacement thickness, when measuring the total resistance of the body to flow around and finding its geometric volume. The analysis shows that for the greater the ratio of the volume of a body to its surface, the more accurate the results of calculating the body's resistance to slow flow around can be obtained based only on the known geometry of the body.

The calculation of the coefficient of resistance to the flow around the body by the shape of an elongated ellipsoid using a simple formula (46) allows us to estimate the minimum value of this coefficient, below which it cannot be in principle. This allows the design to have a reference point of the minimum possible value of the resistance coefficient of the fluid flow around the body.

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