

## Mass and Scattering of a Quantum Particle as a Relativistic Distribution of Matter

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### ABSTRACT

In this paper, a quantum particle is described by a time-space distribution of matter according to the general theory of relativity, which is essentially based on the invariance of the time-space interval. If we consider a time-space interval for two successive points of the trajectory of a punctual quantum particle, as such a particle is conceived in the conventional quantum mechanics, any interaction with a field is described by a variation of this interval. It is remarkable that the velocity induced by such an interaction, for a rest particle is equal to the light velocity, which means that the mass of any punctual particle is null. This is perfectly understandable from the physical point of view, since a particle described only by position coordinates is a punctual one, *i.e.* with a null volume, and, consequently, a null mass. Unlike the present understanding of a quantum particle, as a punctual entity randomly occupying different positions given by its wave function, and with a mass given by coupling with a Higgs boson, we conceive a quantum particle as a distribution of matter, quantized by the equality of the mass given by the space integral of the wave function, with the mass given by the matter dynamics according to the general theory of relativity. In this framework, we previously described the spinor propagation under the interaction with an electromagnetic field, the particle spin, the graviton spin, the black hole dynamics, the matter dynamics of our universe, as a hypersurface defined by the time-space parameters in the total universe, with the Big Bang, inflation, redshift, and the crushing up of the quantum particles at the Big Bang and at the Schwarzschild boundary. Here, we describe essential quantum phenomenon of the particle scattering.

**Keywords:** Quantum particle, wave/group velocity, spinor, Fermi's golden rule, quantum scattering

### INTRODUCTION

We live in an  $N$ -dimensional universe of coordinates [1]

$$(z^n), \quad n=1, 2, \dots, N, \quad (1)$$

we consider as a plane space, with constant metric elements  $h_{mn}$  for a differential interval  $ds$ ,  
 $ds^2 = h_{mn} dz^m dz^n$ . (2)

However, in the total universe, we observe only the time-space, physical coordinates

$$(x^\alpha) = (x^0 = ct, x^1, x^2, x^3) = (x^0 = ct, x^i), \quad i=1, 2, 3. \quad (3)$$

Thus, our objective, physical observations, are reduced to the coordinates of the physical universe, as a hypersurface

$$z^n = z^n(x^\alpha), \quad n = 1, 2, \dots, N-4, \quad (4)$$

in the total universe. The differential interval in the physical universe is of the form

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta, \quad \alpha, \beta = 0, \{i = 1, 2, 3\}, \quad g_{00} > 0, \quad g_{ii} < 0, \quad (5)$$

depending on the metric tensor

$$g_{\alpha\beta} = h_{nm} z^n_{,\alpha} z^m_{,\beta} = z_{n,\alpha} z^n_{,\beta}, \quad n, m = 1, 2, \dots, N-4. \quad (6)$$

In this universe, the various matter structures are composed of various species of particles. A particle of mass  $M$  is conceived as a distribution of matter density, in the coordinate space  $x = (x^1, x^2, x^3)$  and in the momentum space  $p = (p^1, p^2, p^3)$  [2], [3], [4],

$$\begin{aligned} \rho_M(x, t) &= M |\psi(x, t)|^2 \\ \rho_M(p, t) &= M |\varphi(p, t)|^2, \end{aligned} \quad (7)$$

with the amplitudes  $\psi(x, t)$  and  $\varphi(p, t)$ , satisfy the normalization conditions:

$$\int |\psi(x, t)|^2 d^3x = \int |\varphi(p, t)|^2 d^3p = 1. \quad (8)$$

For such a particle in the proper time  $\tau$ , we considered a series expansion in the coordinate space  $\{x^j\}$ , with the relativistic Lagrangian

$$L(p^j, x^j) = -Mc^2 \sqrt{g_{\alpha\beta} v^\alpha v^\beta}, \quad (9)$$

as the coefficient of the time dependent phase, and the wave vector proportional to the momentum  $p^i$ ,

$$\begin{aligned} \psi(x, \tau) &= \frac{1}{(2\pi\hbar)^{3/2}} \int e^{\frac{i}{\hbar}[p^j x^j - L(p, x)\tau]} \varphi(p, \tau) d^3p \\ \varphi(p, \tau) &= \frac{1}{(2\pi\hbar)^{3/2}} \int e^{-\frac{i}{\hbar}[p^j x^j - L(p, x)\tau]} \psi(x, \tau) d^3x. \end{aligned} \quad (10)$$

These wave functions depend on the proper time velocities

$$v^\alpha = \frac{dx^\alpha}{ds} = \frac{dx^\alpha}{cd\tau}, \quad (11)$$

and the momentum

$$p^j = \frac{\partial L}{\partial(c v^j)} = -Mc g_{j\beta} v^\beta. \quad (12)$$

It is remarkable that the wave/group velocity in the coordinate space are equal to the velocity of the amplitude function (10), which we call wave function,

$$\left( \frac{d}{d\tau} x^j \right)_{\text{wave}} = \frac{\partial L}{\partial p^j} = c v^j = \dot{x}^j \quad (13)$$

This means that these waves represent a motion of the Fourier series expansion of the matter density in the coordinate space. At the same time, the wave/group time variations of the momentum,

$$\left( \frac{d}{d\tau} p^j \right)_{\text{wave}} = \frac{d}{d\tau} \frac{\partial L}{\partial \dot{x}^j} = \frac{\partial L}{\partial x^j} = -\frac{1}{2} M c^2 v^\alpha v^\beta g_{\alpha\beta,j}, \quad (14)$$

is of the form of the gradient of a potential. For a classical particle with the kinetic energy

$$T(\vec{p}) = \frac{M \dot{\vec{r}}^2}{2} = \frac{\vec{p}^2}{2M}, \quad (15)$$

in a potential of energy  $U(\vec{r})$ , with the total energy

$$H(\vec{p}, \vec{r}) = T(\vec{p}) + U(\vec{r}), \quad (16)$$

and the Lagrangian

$$L(\vec{p}, \vec{r}) = \vec{p} \dot{\vec{r}} - H(\vec{p}, \vec{r}) = T(\vec{p}) - U(\vec{r}), \quad (17)$$

the wave functions (10) in the local time  $t$  take the form

$$\begin{aligned} \psi(\vec{r}, t) &= \frac{1}{(2\pi\hbar)^{3/2}} \int e^{\frac{i}{\hbar} [\vec{p}\vec{r} - L(\vec{p}, \vec{r})t]} \phi(\vec{p}, t) d^3\vec{p} = \frac{1}{(2\pi\hbar)^{3/2}} \int e^{\frac{i}{\hbar} \{ \vec{p}\vec{r} - [T(\vec{p}) - U(\vec{r})]t \}} \phi(\vec{p}, t) d^3\vec{p} \\ \phi(\vec{p}, t) &= \frac{1}{(2\pi\hbar)^{3/2}} \int e^{-\frac{i}{\hbar} [\vec{p}\vec{r} - L(\vec{p}, \vec{r})t]} \psi(\vec{r}, t) d^3\vec{r} = \frac{1}{(2\pi\hbar)^{3/2}} \int e^{-\frac{i}{\hbar} \{ \vec{p}\vec{r} - [T(\vec{p}) - U(\vec{r})]t \}} \psi(\vec{r}, t) d^3\vec{r}. \end{aligned} \quad (18)$$

We note that the wave/group velocities of these wave packets are of the form of the Hamilton equations

$$\begin{aligned} \left( \frac{d\vec{r}}{dt} \right)_{\text{wave}} &= \frac{\partial T(\vec{p})}{\partial \vec{p}} = \frac{\vec{p}}{M} = \dot{\vec{r}} = \frac{\partial}{\partial \vec{p}} H(\vec{p}, \vec{r}) \\ \left( \frac{d\vec{p}}{dt} \right)_{\text{wave}} &= -\frac{\partial U(\vec{r})}{\partial \vec{r}} = -\frac{\partial}{\partial \vec{r}} H(\vec{p}, \vec{r}). \end{aligned} \quad (19)$$

In the absence of a gravitational field, special relativity, from the additional Lagrangian terms in the time dependent phase of a particle wave function modifying the dynamics of this particle,

with an electric, scalar potential conjugated to time,  $U(\vec{r})$ , and a vector potential, conjugated to the coordinates,  $\vec{A}(\vec{r}, t) \dot{\vec{r}}$ , we reobtained the Maxwell equations and Lorentz's force. In the general theory of relativity, by a tensorial formulation of the electromagnetic field,

$F_{\mu\nu} = A_{\mu,\nu} - A_{\nu,\mu} = A_{\mu;\nu} - A_{\nu;\mu}$  defined by the potential  $A_\mu$ , a unified general theory of gravitation and electromagnetism is obtained [1], [2]. In the relativistic framework, from the general equations (18) with Dirac's Hamiltonian, we obtained spinor propagations for particles and antiparticles, and described the particle spin and the graviton spin dynamics [2], [3]. In these papers, we also described the dynamics of a quantum particle in a black hole. We showed that a black hole is a big quantity of matter, concentrating this matter inside of the Schwarzschild radius, which begins with an explosion-inflation of the central part, sending this matter towards the Schwarzschild boundary, but reaching this boundary only in an infinite time. At the black hole creation, a central quantum particle is crushed up in the momentum space, exploding in the coordinate space. A quantum particle, reaching the Schwarzschild boundary, is crushed up in the coordinate space, in any fluctuation process being returned to this boundary. Conceiving our Universe as a big black hole in the Total Universe, in the framework of the general relativity we obtained a description of major processes, as the Big Bang-inflation with a velocity much larger than the light velocity, the redshift of the light coming from large distances as a gravitational effect, the dark matter as a system of particles with wave functions (18), and the dark energy as the gravitational energy of this black hole with a very large concentration of matter at its Schwarzschild boundary. In [4], we obtained the propagation spinors of a particle-antiparticle system in an electromagnetic field. In this paper, we described two fundamental quantum phenomena. Here we show that in framework of the general relativity, the mass of a punctual quantum particle is null, as it is found in the framework of the Standard Model [5], [6], but, unlike this model, where the mass is obtained by coupling to the Higgs bosons, we obtain the mass as a characteristics of the matter dynamics. We also describe the scattering of a particle by an electromagnetic interaction, as a function of the spin.

### MASS AND INTERACTION OF A QUANTUM PARTICLE WITH A FIELD

In the conventional quantum theory, a particle is a punctual entity, occupying different positions defined by the four time-space coordinates, with a probabilities given by its wave function, as in **figure 1**.

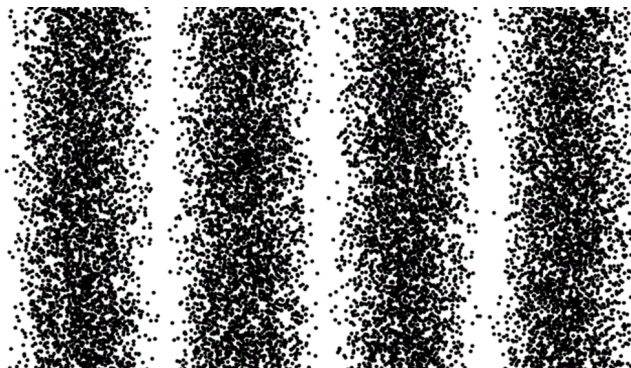


**Figure 1. Quantum particles, hitting a screen with a probability given by a wave function describing a two-slit interference**

In the theory of relativity, which we consider as the fundamental theory of quantum mechanics [2]-[4], the trajectory of a free punctual particle is described by an invariant time-space interval (5) for a flat space,

$$ds = \sqrt{c^2 dt^2 - dx^2} \quad (20)$$

The interaction of a particle with a field is described by a time-space variation  $\delta t \ll dt$ ,  $\delta x \ll cdt$ , leading to a variation  $\delta s$  of the time-space interval

$$\begin{aligned} ds + \delta s &= \sqrt{c^2 (dt + \delta t)^2 - (dx + \delta x)^2} = \sqrt{\underbrace{c^2 dt^2 - dx^2}_{ds^2} + 2c^2 dt \delta t - 2dx \delta x} \\ &= \sqrt{ds^2 + 2c^2 dt \delta t - 2dx \delta x} = ds \sqrt{1 + \frac{2c^2 dt \delta t - 2dx \delta x}{ds^2}} = ds + \frac{c^2 dt \delta t - dx \delta x}{ds} \\ \frac{ds \delta s}{dt \delta t} &= c^2 - \frac{dx}{dt} \cdot \frac{\delta x}{\delta t} \Rightarrow \frac{dx}{dt} \cdot \frac{\delta x}{\delta t} = c^2 - \frac{ds}{dt} \cdot \frac{\delta s}{\delta t} = c^2 - c \sqrt{1 - \frac{dx^2}{c^2 dt^2}} \cdot \frac{\delta s}{\delta x} \cdot \frac{\delta x}{\delta t}. \end{aligned} \quad (21)$$

We obtain an additional

$$\frac{\delta x}{\delta t} = \frac{c}{\frac{dx}{cdt} + \frac{\delta s}{\delta x} \sqrt{1 - \frac{dx^2}{c^2 dt^2}}}, \quad (22)$$

induced by a field to a particle in motion with the velocity  $dx/dt$ . For a particle at rest,

$$\frac{dx}{dt} = 0 \rightarrow \frac{\delta x}{\delta t} = c \frac{\delta x}{\delta s} \rightarrow \frac{\delta s}{\delta t} = c,$$

which means that, by any interaction with a field, a particle reaches the light velocity:

$$\begin{aligned} ds + \delta s &= cdt + c\delta t = \sqrt{c^2 (dt + \delta t)^2 - \delta x^2} = \sqrt{c^2 dt^2 + 2c^2 dt \delta t + c^2 \delta t^2 - \delta x^2} \\ &= cdt \left( 1 + \frac{\delta t}{dt} + \frac{\delta t^2}{2dt^2} - \frac{\delta x^2}{2c^2 dt^2} \right) \Rightarrow \frac{\delta x}{\delta t} = c, \end{aligned} \quad (23)$$

*i.e.* it has a null mass. This is in agreement with the Standard Model, where a quantum particle has a null mass, and gets a mass only by interaction with a field of Higgs bosons [5], [6]. However, our result is the consequence of the punctuality of the particle, simply described by the time-space coordinates, which means no volume – no mass. According to our theory, the mass of a quantum particle is obtained by the description of such a particle as a continuous distribution of matter, in motion according to the general theory of relativity. The dots in **figure 1** do not represent particles reaching different points, as it is generally believe according to the conventional quantum mechanics, but interactions with localized molecular systems of the quantum particles with wave functions occupying the whole space where these systems are placed.

### MATTER DENSITY, THE DENSITY FUNCTION, AND THE DENSITY OPERATOR

According to (18), a particle wave function is of the form of the propagation operator  $e^{\frac{i}{\hbar} \vec{p} \vec{r}}$  depending on coordinates, applied to the time dependent wave function

$$\psi_i(\vec{r}, t) = \frac{1}{(2\pi\hbar)^{3/2}} \int e^{\frac{i}{\hbar} L(\vec{p}, \vec{r})t} \varphi(\vec{p}, t) d^3 \vec{p}. \quad (24)$$

Thus, the matter density (7) is of the form

$$\begin{aligned}\rho_M(\vec{r}, t) &= M |\psi(\vec{r}, t)|^2 = M \psi_t^*(\vec{r}, t) e^{-\frac{i}{\hbar} \vec{p} \vec{r}} e^{\frac{i}{\hbar} \vec{p} \vec{r}} \psi_t(\vec{r}, t) \\ &= M \psi_t^*(\vec{r}, t) \psi_t(\vec{r}, t) = M \rho(\vec{r}, t),\end{aligned}\quad (25)$$

depending of the density function

$$\rho(\vec{r}, t) = \psi_t^*(\vec{r}, t) \psi_t(\vec{r}, t) = \psi_t(\vec{r}, t) \psi_t^*(\vec{r}, t) = \langle \vec{r} | \psi_t \rangle \langle \psi_t | \vec{r} \rangle = \langle \vec{r} | \rho(t) | \vec{r} \rangle, \quad (26)$$

as a diagonal matrix element of the density operator

$$\rho(t) = |\psi_t\rangle \langle \psi_t|. \quad (27)$$

For the Hamiltonian

$$H(\vec{p}, \vec{r}) = \vec{p} \dot{\vec{r}} - L(\vec{p}, \vec{r}), \quad (28)$$

with the Hamilton equations

$$\begin{aligned}\frac{d\vec{r}}{dt} &= \frac{\partial}{\partial \vec{p}} H(\vec{p}, \vec{r}) \\ \frac{d\vec{p}}{dt} &= -\frac{\partial}{\partial \vec{r}} H(\vec{p}, \vec{r}),\end{aligned}\quad (29)$$

one obtains the equation of conservation

$$\frac{d}{dt} H(\vec{p}, \vec{r}) = \frac{\partial H}{\partial \vec{p}} \frac{d\vec{p}}{dt} + \frac{\partial H}{\partial \vec{r}} \frac{d\vec{r}}{dt} = -\frac{\partial H}{\partial \vec{p}} \frac{\partial H}{\partial \vec{r}} + \frac{\partial H}{\partial \vec{r}} \frac{\partial H}{\partial \vec{p}} = 0, \quad (30)$$

with the energy as an eigenvalue,

$$H(\vec{p}, \vec{r}) = E. \quad (31)$$

With the Hamiltonian (28) as a conservative function, the time dependent wave function (24) takes the form

$$\psi_t(\vec{r}, t) = \frac{1}{(2\pi\hbar)^{3/2}} \int e^{-\frac{i}{\hbar} [\vec{p} \dot{\vec{r}} - H(\vec{p}, \vec{r})] t} \varphi(\vec{p}, t) d^3 \vec{p} \quad (32)$$

By neglecting the explicit time dependence of the matter distribution function in the dynamics of a quantum particle,

$$\frac{\partial}{\partial t} \varphi(\vec{p}, t) = 0, \quad (33)$$

the wave function (32), is obtained as a solution of the dynamic equation

$$\frac{\partial}{\partial t} \psi_t(\vec{r}, t) = -\frac{i}{\hbar} [\vec{p} \dot{\vec{r}} - H(\vec{p}, \vec{r})] \psi_t(\vec{r}, t). \quad (34)$$

This equation, with the wave function  $\psi_t(\vec{r}, t)$  as the projection of a state  $|\psi_t(t)\rangle$  on an eigenstate  $|\vec{r}\rangle$ ,

$$\frac{\partial}{\partial t} \langle \vec{r} | \psi_t(t) \rangle = -\frac{i}{\hbar} [\vec{p} \dot{\vec{r}} - H(\vec{p}, \vec{r})] \langle \vec{r} | \psi_t(t) \rangle, \quad (35)$$

and the representations of an arbitrary operator  $A$  with the matrix elements

$$\begin{aligned} \langle \phi(t) | A | \psi(t) \rangle &= \int \langle \phi(t) | \vec{r} \rangle A \langle \vec{r} | \psi(t) \rangle d^3\vec{r} = \langle \phi(t) | 1A | \psi(t) \rangle = \langle \phi(t) | \int |\vec{r}\rangle d^3\vec{r} \langle \vec{r} | A | \psi(t) \rangle \\ &= \int \langle \phi(t) | \vec{r} \rangle d^3\vec{r} \langle \vec{r} | A | \psi(t) \rangle, \end{aligned}$$

takes the form

$$\langle \vec{r} | \frac{\partial}{\partial t} | \psi_t(t) \rangle = -\frac{i}{\hbar} \langle \vec{r} | [\vec{p} \dot{\vec{r}} - H(\vec{p}, \vec{r})] | \psi_t(t) \rangle. \quad (36)$$

We obtain the dynamic equation for a particle state:

$$\frac{\partial}{\partial t} | \psi_t(t) \rangle = -\frac{i}{\hbar} L(\vec{p}, \vec{r}) | \psi_t(t) \rangle = -\frac{i}{\hbar} [\vec{p} \dot{\vec{r}} - H(\vec{p}, \vec{r})] | \psi_t(t) \rangle. \quad (37)$$

With this equation for the density operator (27),

$$\begin{aligned} \frac{\partial}{\partial t} \rho(t) &= \frac{\partial}{\partial t} | \psi_t(t) \rangle \langle \psi_t(t) | + | \psi_t(t) \rangle \frac{\partial}{\partial t} \langle \psi_t(t) | \\ &= -\frac{i}{\hbar} L(\vec{p}, \vec{r}) | \psi_t(t) \rangle \langle \psi_t(t) | + \frac{i}{\hbar} | \psi_t(t) \rangle \langle \psi_t(t) | L(\vec{p}, \vec{r}), \end{aligned}$$

we obtain the dynamic equation for the this operator:

$$\frac{\partial}{\partial t} \rho(t) = -\frac{i}{\hbar} [L(\vec{p}, \vec{r}), \rho(t)] = -\frac{i}{\hbar} [\vec{p} \dot{\vec{r}} - H(\vec{p}, \vec{r}), \rho(t)]. \quad (38)$$

### QUANTUM DYNAMICS AND MATTER CONSERVATION

The dynamics of a quantum particle with an energy  $H(\vec{p}, \vec{r}) = E$  is described by equation (34), where we consider the explicit expression of the momentum operator,

$$\frac{\partial}{\partial t} \psi_t(\vec{r}, t) = -\frac{i}{\hbar} \left( -i\hbar \dot{\vec{r}} \frac{\partial}{\partial \vec{r}} - E \right) \psi_t(\vec{r}, t). \quad (39)$$

With the complex conjugate of this equation,

$$\frac{\partial}{\partial t} \psi_t^*(\vec{r}, t) = \frac{i}{\hbar} \left( i\hbar \dot{\vec{r}} \frac{\partial}{\partial \vec{r}} - E \right) \psi_t^*(\vec{r}, t),$$

we obtain the dynamic equation of the density function,

$$\begin{aligned} \frac{\partial}{\partial t} \rho(\vec{r}, t) &= \frac{\partial}{\partial t} \psi_t^*(\vec{r}, t) \psi_t(\vec{r}, t) + \psi_t^*(\vec{r}, t) \frac{\partial}{\partial t} \psi_t(\vec{r}, t) \\ &= \frac{i}{\hbar} \left( i\hbar \dot{\vec{r}} \frac{\partial}{\partial \vec{r}} - E \right) \psi_t^*(\vec{r}, t) \psi_t(\vec{r}, t) - \psi_t^*(\vec{r}, t) \frac{i}{\hbar} \left( -i\hbar \dot{\vec{r}} \frac{\partial}{\partial \vec{r}} - E \right) \psi_t(\vec{r}, t) \\ &= -\dot{\vec{r}} \frac{\partial}{\partial \vec{r}} \rho(\vec{r}, t). \end{aligned}$$

Thus, for the matter density of a quantum particle, we obtain a null total derivative

$$\frac{d}{dt} \rho(\vec{r}, t) = \frac{\partial}{\partial t} \rho(\vec{r}, t) + \dot{\vec{r}} \cdot \frac{\partial}{\partial \vec{r}} \rho(\vec{r}, t) = 0 \quad (40)$$

This is an equation of conservation,

$$\frac{\partial}{\partial t} \int_V \rho(\vec{r}, t) d^3\vec{r} = - \oint_{\Sigma_V} \rho(\vec{r}, t) \dot{\vec{r}} d^2\vec{r} = - \oint_{\Sigma_V} \vec{J}(\vec{r}, \dot{\vec{r}}, t) d^2\vec{r} \quad (41)$$

where the mass variation in an arbitrary volume  $V$ , as an integral of the density  $\rho(\vec{r}, t)$  over this volume, is given by the matter flow through the surface  $\Sigma_V$  of this volume, with the flux density

$$\vec{J}(\vec{r}, \dot{\vec{r}}, t) = \rho(\vec{r}, t) \dot{\vec{r}} \quad (42)$$

We notice that, unlike the Schrödinger dynamic equation, which describes only an explicit dependence on time induced by the particle Hamiltonian, according to the dynamic equation (34) the Hamiltonian induces a total time variation of the wave function,

$$\frac{d}{dt} \psi_t(\vec{r}, t) = \left( \frac{\partial}{\partial t} + \dot{\vec{r}} \cdot \frac{\partial}{\partial \vec{r}} \right) \psi_t(\vec{r}, t) = \frac{i}{\hbar} H(\vec{p}, \vec{r}) \psi_t(\vec{r}, t) \quad (43)$$

which is in agreement with the mass conservation. From this equation for a nonrelativistic particle in a potential  $U(\vec{r})$ ,

$$\begin{aligned} \frac{\partial}{\partial t} \psi_t(\vec{r}, t) &= \left[ -\dot{\vec{r}} \cdot \frac{\partial}{\partial \vec{r}} - \frac{i\hbar}{2M} \frac{\partial^2}{\partial \vec{r}^2} + \frac{i}{\hbar} U(\vec{r}) \right] \psi_t(\vec{r}, t) \quad / \overline{\psi_t^*(\vec{r}, t)} \\ \frac{\partial}{\partial t} \psi_t^*(\vec{r}, t) &= \left[ -\dot{\vec{r}} \cdot \frac{\partial}{\partial \vec{r}} + \frac{i\hbar}{2M} \frac{\partial^2}{\partial \vec{r}^2} - \frac{i}{\hbar} U(\vec{r}) \right] \psi_t^*(\vec{r}, t) \quad / \overline{\psi_t(\vec{r}, t)}, \end{aligned}$$

we obtain again an equation of conservation:

$$\begin{aligned} \frac{\partial}{\partial t} \int_V \rho(\vec{r}, t) d^3\vec{r} &= \frac{\partial}{\partial t} \int_V \psi_t^*(\vec{r}, t) \psi_t(\vec{r}, t) d^3\vec{r} \\ &= \int_V \left[ \psi_t(\vec{r}, t) \frac{\partial}{\partial t} \psi_t^*(\vec{r}, t) + \psi_t^*(\vec{r}, t) \frac{\partial}{\partial t} \psi_t(\vec{r}, t) \right] d^3\vec{r} \\ &= \int_V \left\{ -\dot{\vec{r}} \cdot \frac{\partial}{\partial \vec{r}} [\psi_t^*(\vec{r}, t) \psi_t(\vec{r}, t)] - \frac{i\hbar}{2M} \frac{\partial}{\partial \vec{r}} \left[ \psi_t^*(\vec{r}, t) \frac{\partial}{\partial \vec{r}} \psi_t(\vec{r}, t) - \psi_t(\vec{r}, t) \frac{\partial}{\partial \vec{r}} \psi_t^*(\vec{r}, t) \right] \right\} d^3\vec{r} \\ &= \oint_{\Sigma_V} \left\{ -\dot{\vec{r}} \psi_t^*(\vec{r}, t) \psi_t(\vec{r}, t) - \frac{i\hbar}{2M} \left[ \psi_t^*(\vec{r}, t) \frac{\partial}{\partial \vec{r}} \psi_t(\vec{r}, t) - \psi_t(\vec{r}, t) \frac{\partial}{\partial \vec{r}} \psi_t^*(\vec{r}, t) \right] \right\} d^2\vec{r} \\ &= \oint_{\Sigma_V} \left\{ -\dot{\vec{r}} \rho(\vec{r}, t) - \frac{i\hbar}{2M} \left[ \psi_t^*(\vec{r}, t) \frac{\partial}{\partial \vec{r}} \psi_t(\vec{r}, t) - \psi_t(\vec{r}, t) \frac{\partial}{\partial \vec{r}} \psi_t^*(\vec{r}, t) \right] \right\} d^2\vec{r}. \end{aligned}$$

This equation

$$\frac{\partial}{\partial t} \int_V \rho(\vec{r}, t) d^3\vec{r} = - \oint_{\Sigma_V} \vec{J}_\psi(\vec{r}, \dot{\vec{r}}, t) d^2\vec{r}, \quad (44)$$

depends on the quantum matter flow density

$$\vec{J}_\psi(\vec{r}, \dot{\vec{r}}, t) = \dot{\vec{r}} \rho(\vec{r}, t) + \frac{i\hbar}{2M} \left[ \psi_t^*(\vec{r}, t) \frac{\partial}{\partial \vec{r}} \psi_t(\vec{r}, t) - \psi_t(\vec{r}, t) \frac{\partial}{\partial \vec{r}} \psi_t^*(\vec{r}, t) \right], \quad (45)$$

which, besides the drift term proportional to the velocity, includes a diffusion term, proportional to the gradient of the wave function.

### FERMI'S GOLDEN RULE FOR QUANTUM PARTICLES AS DISTRIBUTIONS OF MATTER

We consider a scattering/tunneling process of a quantum particle with a Hamiltonian  $H_0(\vec{p}, \vec{r})$ , described by a perturbing potential  $V$  [7]. Such a process is described by the dynamic equation (38),

$$\frac{\partial}{\partial t} \rho(t) = -\frac{i}{\hbar} [\vec{p} \dot{\vec{r}} - H_0(\vec{p}, \vec{r}) - V, \rho(t)] = -\frac{i}{\hbar} [L_0(\vec{p}, \vec{r}) - V, \rho(t)], \quad (46)$$

which in the interaction picture, with a perturbation strength coefficient  $\varepsilon$ , is of the form

$$\frac{\partial}{\partial t} \tilde{\rho}(t) = \frac{i}{\hbar} [\varepsilon \tilde{V}(t), \tilde{\rho}(t)], \quad (47)$$

for the interaction picture operators

$$\begin{aligned} \tilde{\rho}(t) &= e^{\frac{i}{\hbar} L_0(\vec{p}, \vec{r})t} \rho(t) e^{-\frac{i}{\hbar} L_0(\vec{p}, \vec{r})t} \\ \tilde{V}(t) &= e^{\frac{i}{\hbar} L_0(\vec{p}, \vec{r})t} V e^{-\frac{i}{\hbar} L_0(\vec{p}, \vec{r})t}. \end{aligned} \quad (48)$$

According to the theory of perturbations, with a series expansion

$$\tilde{\rho}(t) = \tilde{\rho}^{(0)}(t) + \varepsilon \tilde{\rho}^{(1)}(t) + \varepsilon^2 \tilde{\rho}^{(2)}(t) + \dots, \quad (49)$$

from the dynamic equation (47),

$$\frac{\partial}{\partial t} [\tilde{\rho}^{(0)}(t) + \varepsilon \tilde{\rho}^{(1)}(t) + \varepsilon^2 \tilde{\rho}^{(2)}(t) + \dots] = \frac{i}{\hbar} [\varepsilon \tilde{V}(t), \tilde{\rho}^{(0)}(t) + \varepsilon \tilde{\rho}^{(1)}(t) + \varepsilon^2 \tilde{\rho}^{(2)}(t) + \dots],$$

we obtain the system of equations:

$$\begin{aligned} \frac{\partial}{\partial t} \tilde{\rho}^{(0)}(t) &= 0, \quad \frac{\partial}{\partial t} \tilde{\rho}^{(1)}(t) = \frac{i}{\hbar} [\tilde{V}(t), \tilde{\rho}^{(0)}(t)], \quad \frac{\partial}{\partial t} \tilde{\rho}^{(2)}(t) = \frac{i}{\hbar} [\tilde{V}(t), \tilde{\rho}^{(1)}(t)], \\ &\dots \end{aligned} \quad (50)$$

From these equations, we obtain a time independent zeroth order term,

$$\tilde{\rho}^{(0)}(t) = |0\rangle\langle 0|, \quad (51)$$

a time dependent first order term

$$\tilde{\rho}^{(1)}(t) = \frac{i}{\hbar} \int_0^t [\tilde{V}(t'), \tilde{\rho}^{(0)}] dt' \quad (52)$$

and the time derivative of the second order term as a function of the time independent zeroth order term, and the time dependent perturbing potential,

$$\frac{\partial}{\partial t} \tilde{\rho}^{(2)}(t) = \frac{i}{\hbar} [\tilde{V}(t), \tilde{\rho}^{(1)}(t)] = -\frac{1}{\hbar^2} \int_0^t [\tilde{V}(t), [\tilde{V}(t'), \tilde{\rho}^{(0)}]] dt' \quad (53)$$

From this equation, we obtain scattering/tunneling rates, as the time derivatives of the diagonal elements of the density matrix in the space of the final state, which depend on the transition matrix elements of the scattering/tunneling operator:

$$\Gamma_i = \frac{\partial}{\partial t} \langle i | \tilde{\rho}(t) | i \rangle = \frac{1}{\hbar^2} \int_0^t [\tilde{V}_{i0}(t) \tilde{V}_{0i}(t') + \tilde{V}_{i0}(t') \tilde{V}_{0i}(t)] dt' \quad (54)$$

From (48), we obtain the these matrix elements,

$$\begin{aligned} \tilde{V}_{i0}(t) &= \langle i | \tilde{V}(t) | 0 \rangle = e^{\frac{i}{\hbar}(L_i - L_0)t} V_{i0}, & \tilde{V}_{0i}(t') &= \langle 0 | \tilde{V}(t') | i \rangle = e^{-\frac{i}{\hbar}(L_i - L_0)t'} V_{0i} \\ \tilde{V}_{i0}(t') &= \langle i | \tilde{V}(t') | 0 \rangle = e^{\frac{i}{\hbar}(L_i - L_0)t'} V_{i0}, & \tilde{V}_{0i}(t) &= \langle 0 | \tilde{V}(t) | i \rangle = e^{-\frac{i}{\hbar}(L_i - L_0)t} V_{0i} \end{aligned} \quad (55)$$

as functions of the frequencies

$$\omega_i = \frac{L_i - L_0}{\hbar} \quad (56)$$

With these expressions, the decay rate (54) of a transition  $|0\rangle \rightarrow |i\rangle$  is

$$\begin{aligned} \Gamma_i &= \frac{\partial}{\partial t} \tilde{\rho}_{ii}(t) = \frac{1}{\hbar^2} \int_0^t |V_{i0}|^2 \left( e^{i\omega_i(t-t')} + e^{-i\omega_i(t-t')} \right) dt' = \frac{2}{\hbar^2} \int_0^t |V_{i0}|^2 \cos[\omega_i(t-t')] dt' \\ &= -\frac{2}{\hbar^2} |V_{i0}|^2 \frac{\sin[\omega_i(t-t')]}{\omega_i} \Big|_0^t = \frac{2}{\hbar^2} |V_{i0}|^2 \frac{\sin(\omega_i t)}{\omega_i} \end{aligned} \quad (57)$$

Fermi's golden rule refer to the decay rate in a quasi-continuum of states  $|i\rangle$  with the density  $g(\omega_i)$ ,

$$\Gamma = \frac{\partial}{\partial t} \sum_i \tilde{\rho}_{ii}(t) = \frac{2}{\hbar^2} |V_{i0}|^2 g(\omega_i) \int_{-\infty}^{\infty} \frac{\sin(\omega_i t)}{\omega_i} d\omega_i = \frac{2\pi}{\hbar^2} |V_{i0}|^2 g(\omega_i) \quad (58)$$

In the nonrelativistic case, for a particle in a potential  $U(\vec{r})$ , in an initial time dependent state  $\psi_{i0}(\vec{r}, t)$  of energy  $E_0$ , under the action of a scattering/tunneling operator  $V$  [7], which sends this particle in a final state  $\psi_{ii}(\vec{r}, t)$

$$\begin{aligned}
 (H_0 + V)\psi_{i0}(\vec{r}, t) &= \left[ -\frac{\hbar^2}{2M} \frac{\partial^2}{\partial \vec{r}^2} + U(\vec{r}) + V \right] \psi_{i0}(\vec{r}, t) = E_0 \psi_{i0}(\vec{r}, t) \quad / \quad \overline{\psi_{ii}^*(\vec{r}, t)} \\
 H_0 \psi_{ii}^*(\vec{r}, t) &= \left[ -\frac{\hbar^2}{2M} \frac{\partial^2}{\partial \vec{r}^2} + U(\vec{r}) \right] \psi_{ii}^*(\vec{r}, t) = E_i \psi_{ii}^*(\vec{r}, t) \quad / \quad \overline{\psi_{i0}(\vec{r}, t)}
 \end{aligned} \quad (59)$$

by integration over the volume  $\Omega$  of the initial state, we obtain the matrix element as an integral over the surface  $\Sigma_\Omega$  of this volume:

$$\begin{aligned}
 V_{i0} &= \frac{\hbar^2}{2M} \int_\Omega \left[ \psi_{ii}^*(\vec{r}, t) \frac{\partial^2}{\partial \vec{r}^2} \psi_{i0}(\vec{r}, t) - \psi_{i0}(\vec{r}, t) \frac{\partial^2}{\partial \vec{r}^2} \psi_{ii}^*(\vec{r}, t) \right] d^3\vec{r} \\
 &= \frac{\hbar^2}{2M} \oint_{\Sigma_\Omega} \left[ \psi_{ii}^*(\vec{r}, t) \frac{\partial}{\partial \vec{r}} \psi_{i0}(\vec{r}, t) - \psi_{i0}(\vec{r}, t) \frac{\partial}{\partial \vec{r}} \psi_{ii}^*(\vec{r}, t) \right] d^2\vec{r}.
 \end{aligned} \quad (60)$$

### SCATTERING/TUNNELLING RATE IN AN ELECTROMAGNETIC FIELD

In the relativistic rate (58), Fermi's golden rule, the relativistic effects are entirely described by the relativistic wave functions in the matrix elements  $V_{i0}$ , which, for a particle in electromagnetic field [4], are functions of the canonical momentum

$$\vec{P} = \vec{p}(\vec{r}, t) + e\vec{A}(\vec{r}, t), \quad (61)$$

depending on the mechanical momentum  $\vec{p}(\vec{r}, t)$ , the electric charge  $e$ , and the vector potential  $\vec{A}(\vec{r}, t)$ . With the width  $L_P$ , of a particle occupying the domain  $\Delta^3\vec{P}$  in the momentum space, which gives finite dimensions of this particle in the coordinate space, (187) in [4],

$$L_P^3 = \int_{\Delta^3\vec{P}} \sqrt{1 + \frac{\vec{P}^2}{m^2}} d^3\vec{P}, \quad (62)$$

these wave functions are of the form (183) in [4],

$$\psi_i(\vec{r}, t) = \frac{1}{(2\pi\hbar L_P)^{3/2}} \int e^{\frac{i}{\hbar} [\vec{P}\vec{r} - (\alpha_0 Mc^2 + c\alpha_j P_j)] t} e^{-\frac{i}{\hbar} \vec{P}\vec{r}} u(\vec{P}) d^3\vec{P}, \quad (63)$$

where  $\alpha_0, \alpha_j$  are Dirac's spin matrices. Here,

$$u(\vec{P}) = \sqrt{\frac{\tilde{E} + m}{2m}} \begin{pmatrix} \tilde{u}(\vec{P}) \\ \frac{\vec{\sigma}\vec{P}}{\tilde{E} + m} \tilde{u}(\vec{P}) \end{pmatrix} \quad (64)$$

are the particle four-dimensional spinors of the form (176) in [4], with the coefficient (178) in [4], depending on the two-dimensional Pauli spin matrices  $\sigma_i$ ,  $\vec{\sigma} = \vec{1}_i \sigma_i$ , the rest momentum  $m = Mc$ , and the relativistic particle-field momentum (171) in [4],

$$\tilde{E}(\vec{P}) = \sqrt{m^2 + \vec{P}^2}, \quad (65)$$

which is the relativistic factor of the relativistic energy  $E(\vec{P}) = c\tilde{E}(\vec{P})$ , as  $\tilde{u}(\vec{P})$  is the two-dimensional spinor, with the eigenvalues

$$\tilde{u} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \tilde{u}^\dagger \tilde{u} = 1 \quad (66)$$

In the relativistic case, for a particle in an initial time dependent state  $\psi_{i0}(\vec{r}, t)$ , under the action of a scattering/tunneling operator  $V$ , which sends this particle in a final state  $\psi_{if}(\vec{r}, t)$ , the scattering process is described by a system of equations similar to (59), but with solutions of the form (63), (131) in [4],

$$\begin{aligned} \left[ i\hbar \left( \frac{\partial}{\partial t} + \alpha_j c \frac{\partial}{\partial x^j} \right) + \alpha_0 Mc^2 + \alpha_0 V - \vec{P} \vec{\alpha} \right] \psi_{i0}(\vec{r}, t) &= 0 \quad / \overline{\psi_{if}^\dagger(\vec{r}, t) \alpha_0} \\ \left[ -i\hbar \left( \frac{\partial}{\partial t} + \alpha_j c \frac{\partial}{\partial x^j} \right) + \alpha_0 Mc^2 - \vec{P} \vec{\alpha} \right] \psi_{if}^\dagger(\vec{r}, t) &= 0 \quad / \overline{\alpha_0 \psi_{i0}(\vec{r}, t)}. \end{aligned} \quad (67)$$

By multiplying these equation to left with  $\psi_{if}^\dagger(\vec{r}, t) \alpha_0$  and, respectively, with  $\alpha_0 \psi_{i0}(\vec{r}, t)$  to right,

$$\begin{aligned} i\hbar \psi_{if}^\dagger(\vec{r}, t) \alpha_0 \frac{\partial}{\partial t} \psi_{i0}(\vec{r}, t) + i\hbar c \psi_{if}^\dagger(\vec{r}, t) \alpha_0 \alpha_j \frac{\partial}{\partial x^j} \psi_{i0}(\vec{r}, t) + \overline{Mc^2 \psi_{if}^\dagger(\vec{r}, t) \alpha_0 \alpha_0 \psi_{i0}(\vec{r}, t)} \\ + \psi_{if}^\dagger(\vec{r}, t) \underbrace{\alpha_0 \alpha_0}_1 V \psi_{i0}(\vec{r}, t) - \overline{\vec{P} \vec{\alpha} \psi_{if}^\dagger(\vec{r}, t) \alpha_0 \psi_{i0}(\vec{r}, t)} = 0 \\ -i\hbar \frac{\partial}{\partial t} \psi_{if}^\dagger(\vec{r}, t) \alpha_0 \psi_{i0}(\vec{r}, t) - i\hbar c \frac{\partial}{\partial x^j} \psi_{if}^\dagger(\vec{r}, t) \alpha_j \alpha_0 \psi_{i0}(\vec{r}, t) + \overline{Mc^2 \psi_{if}^\dagger(\vec{r}, t) \alpha_0 \alpha_0 \psi_{i0}(\vec{r}, t)} \\ - \overline{\vec{P} \vec{\alpha} \psi_{if}^\dagger(\vec{r}, t) \alpha_0 \psi_{i0}(\vec{r}, t)} = 0, \end{aligned}$$

we obtain the expression

$$\psi_{if}^\dagger(\vec{r}, t) V \psi_{i0}(\vec{r}, t) = -i\hbar \frac{\partial}{\partial t} [\psi_{if}^\dagger(\vec{r}, t) \gamma^0 \psi_{i0}(\vec{r}, t)] - i\hbar c \frac{\partial}{\partial x^j} [\psi_{if}^\dagger(\vec{r}, t) \gamma^j \psi_{i0}(\vec{r}, t)], \quad (68)$$

as a function of the Dirac matrices  $\gamma^0 = \alpha_0$ ,  $\gamma^j = \alpha_0 \alpha_j$ , which satisfy the Clifford algebra [4]. For the first term of this equation, with the wave functions of the form (63), we obtain the expression

$$\begin{aligned}
& \psi_{ii}^\dagger(\vec{r}, t) \gamma^0 \psi_{i0}(\vec{r}, t) \\
&= \frac{1}{(2\pi\hbar L_p)^3} \int u^\dagger(\vec{P}) e^{\frac{i}{\hbar}\vec{P}\vec{r}} e^{\frac{i}{\hbar}[\vec{P}\vec{r} - (\alpha_0 M c^2 + c\alpha_j P_j)]t} \gamma^0 e^{-\frac{i}{\hbar}[\vec{P}'\vec{r} - (\alpha_0 M c^2 + c\alpha_j P'_j)]t} e^{-\frac{i}{\hbar}\vec{P}'\vec{r}} u(\vec{P}') d^3\vec{P} d^3\vec{P}' \\
&= \frac{1}{(2\pi\hbar L_p)^3} \int u^\dagger(\vec{P}) e^{\frac{i}{\hbar}\vec{P}\vec{r}} e^{\frac{i}{\hbar}[\vec{P}\vec{r} - c\sqrt{M^2 c^2 + \vec{P}^2}]t} \gamma^0 e^{-\frac{i}{\hbar}[\vec{P}'\vec{r} - c\sqrt{M^2 c^2 + \vec{P}'^2}]t} e^{-\frac{i}{\hbar}\vec{P}'\vec{r}} u(\vec{P}') d^3\vec{P} d^3\vec{P}' \\
&= \frac{1}{(2\pi\hbar L_p)^3} \int e^{\frac{i}{\hbar}(\vec{P}-\vec{P}')\vec{r}} e^{\frac{i}{\hbar}[(\vec{P}-\vec{P}')\vec{r} - c(\sqrt{M^2 c^2 + \vec{P}^2} - \sqrt{M^2 c^2 + \vec{P}'^2})]t} u^\dagger(\vec{P}) \gamma^0 u(\vec{P}') d^3\vec{P} d^3\vec{P}'.
\end{aligned}$$

For simplicity, in the following we take the spinor (64) with the mean value of the coefficient  $\tilde{E}(\vec{P}) + m$ . With this spinor, and

$$\gamma^0 = \begin{pmatrix} \hat{1} & 0 \\ 0 & -\hat{1} \end{pmatrix}, \quad (69)$$

we obtain

$$\begin{aligned}
\psi_{ii}^\dagger(\vec{r}, t) \gamma^0 \psi_{i0}(\vec{r}, t) &= \frac{1}{(2\pi\hbar L_p)^3} \frac{\tilde{E} + m}{2m} \int e^{\frac{i}{\hbar}(\vec{P}-\vec{P}')\vec{r}} e^{\frac{i}{\hbar}[(\vec{P}-\vec{P}')\vec{r} - c(\sqrt{M^2 c^2 + \vec{P}^2} - \sqrt{M^2 c^2 + \vec{P}'^2})]t} \\
&\quad \left( \tilde{u}^\dagger(\vec{P}) \quad \tilde{u}^\dagger(\vec{P}) \frac{\vec{\sigma}\vec{P}}{\tilde{E} + m} \right) \begin{pmatrix} \hat{1} & 0 \\ 0 & -\hat{1} \end{pmatrix} \begin{pmatrix} \tilde{u}(\vec{P}') \\ \frac{\vec{\sigma}\vec{P}'}{\tilde{E} + m} \tilde{u}(\vec{P}') \end{pmatrix} d^3\vec{P} d^3\vec{P}' \\
&= \frac{1}{(2\pi\hbar L_p)^3} \frac{\tilde{E} + m}{2m} \int e^{\frac{i}{\hbar}(\vec{P}-\vec{P}')\vec{r}} e^{\frac{i}{\hbar}[(\vec{P}-\vec{P}')\vec{r} - c(\sqrt{M^2 c^2 + \vec{P}^2} - \sqrt{M^2 c^2 + \vec{P}'^2})]t} \\
&\quad \left( \tilde{u}^\dagger(\vec{P}) \quad \tilde{u}^\dagger(\vec{P}) \frac{\vec{\sigma}\vec{P}}{\tilde{E} + m} \right) \begin{pmatrix} \tilde{u}(\vec{P}') \\ -\frac{\vec{\sigma}\vec{P}'}{\tilde{E} + m} \tilde{u}(\vec{P}') \end{pmatrix} d^3\vec{P} d^3\vec{P}'.
\end{aligned}$$

With the formula

$$(\vec{\sigma}\vec{A})(\vec{\sigma}\vec{B}) = \vec{A}\vec{B} + i\vec{\sigma}(\vec{A} \times \vec{B}), \quad (70)$$

this expression takes the form

$$\begin{aligned}
\psi_{ii}^\dagger(\vec{r}, t) \gamma^0 \psi_{i0}(\vec{r}, t) &= \frac{1}{(2\pi\hbar L_p)^3} \frac{\tilde{E} + m}{2m} \int e^{\frac{i}{\hbar}(\vec{P}-\vec{P}')\vec{r}} e^{\frac{i}{\hbar}[(\vec{P}-\vec{P}')\vec{r} - c(\sqrt{M^2 c^2 + \vec{P}^2} - \sqrt{M^2 c^2 + \vec{P}'^2})]t} \\
&\quad \left[ 1 - \frac{\vec{P}\vec{P}' + i\vec{\sigma}(\vec{P} \times \vec{P}')}{(\tilde{E} + m)^2} \right] \tilde{u}^\dagger(\vec{P}) \tilde{u}(\vec{P}') d^3\vec{P} d^3\vec{P}'.
\end{aligned} \quad (71)$$

For the second term of equation (68), we consider the expression

$$\begin{aligned} & \psi_{i0}^\dagger(\vec{r}, t) \gamma^j \psi_{i0}(\vec{r}, t) \\ &= \frac{1}{(2\pi\hbar L_p)^3} \int u^\dagger(\vec{P}) e^{\frac{i}{\hbar} \vec{P} \vec{r}} e^{\frac{i}{\hbar} [\vec{P} \vec{r} - (\alpha_0 M c^2 + c \alpha_j P_j)] t} \gamma^j e^{-\frac{i}{\hbar} [\vec{P}' \vec{r} - (\alpha_0 M c^2 + c \alpha_j P'_j)] t} e^{-\frac{i}{\hbar} \vec{P}' \vec{r}} u(\vec{P}') d^3 \vec{P} d^3 \vec{P}' \\ &= \frac{1}{(2\pi\hbar L_p)^3} \int e^{\frac{i}{\hbar} (\vec{P} - \vec{P}') \vec{r}} e^{\frac{i}{\hbar} [(\vec{P} - \vec{P}') \vec{r} - c(\sqrt{M^2 c^2 + \vec{P}^2} - \sqrt{M^2 c^2 + \vec{P}'^2})] t} u^\dagger(\vec{P}) \gamma^j u(\vec{P}') d^3 \vec{P} d^3 \vec{P}', \end{aligned}$$

which, with (64), and the expressions of the Dirac matrices as functions of the Pauli spin matrices,

$$\gamma^j = \begin{pmatrix} 0 & \sigma_j \\ -\sigma_j & 0 \end{pmatrix}, \quad (72)$$

takes the form

$$\begin{aligned} \psi_{i0}^\dagger(\vec{r}, t) \gamma^j \psi_{i0}(\vec{r}, t) &= \frac{1}{(2\pi\hbar L_p)^3} \frac{\tilde{E} + m}{2m} \int e^{\frac{i}{\hbar} (\vec{P} - \vec{P}') \vec{r}} e^{\frac{i}{\hbar} [(\vec{P} - \vec{P}') \vec{r} - c(\sqrt{M^2 c^2 + \vec{P}^2} - \sqrt{M^2 c^2 + \vec{P}'^2})] t} \\ &\quad \begin{pmatrix} \tilde{u}_+^\dagger(\vec{P}) & \tilde{u}_+^\dagger(\vec{P}) \frac{\vec{\sigma} \vec{P}}{\tilde{E} + m} \end{pmatrix} \begin{pmatrix} 0 & \sigma_j \\ -\sigma_j & 0 \end{pmatrix} \begin{pmatrix} \tilde{u}_+(\vec{P}') \\ \frac{\vec{\sigma} \vec{P}'}{\tilde{E} + m} \tilde{u}_+(\vec{P}') \end{pmatrix} d^3 \vec{P} d^3 \vec{P}' \\ &= \frac{1}{(2\pi\hbar L_p)^3} \frac{\tilde{E} + m}{2m} \int e^{\frac{i}{\hbar} (\vec{P} - \vec{P}') \vec{r}} e^{\frac{i}{\hbar} [(\vec{P} - \vec{P}') \vec{r} - c(\sqrt{M^2 c^2 + \vec{P}^2} - \sqrt{M^2 c^2 + \vec{P}'^2})] t} \\ &\quad \begin{pmatrix} \tilde{u}_+^\dagger(\vec{P}) & \tilde{u}_+^\dagger(\vec{P}) \frac{\vec{\sigma} \vec{P}}{\tilde{E} + m} \end{pmatrix} \begin{pmatrix} \frac{\sigma_j \sigma_k P'^k}{\tilde{E} + m} \tilde{u}_+(\vec{P}') \\ -\sigma_j \tilde{u}_+(\vec{P}') \end{pmatrix} d^3 \vec{P} d^3 \vec{P}'. \end{aligned}$$

With the expression

$$\sigma_i = i \varepsilon_{ijk} \sigma_j \sigma_k, \quad (73)$$

where  $\varepsilon_{ijk}$  is the Levi-Civita symbol, we obtain

$$\begin{aligned} \psi_{i0}^\dagger(\vec{r}, t) \gamma^j \psi_{i0}(\vec{r}, t) &= \frac{1}{(2\pi\hbar L_p)^3} \frac{\tilde{E} + m}{2m} \int e^{\frac{i}{\hbar} (\vec{P} - \vec{P}') \vec{r}} e^{\frac{i}{\hbar} [(\vec{P} - \vec{P}') \vec{r} - c(\sqrt{M^2 c^2 + \vec{P}^2} - \sqrt{M^2 c^2 + \vec{P}'^2})] t} \\ &\quad \left( \tilde{u}_+^\dagger(\vec{P}) \frac{\sigma_j \sigma_k P'^k}{\tilde{E} + m} \tilde{u}_+(\vec{P}') - \tilde{u}_+^\dagger(\vec{P}) \frac{\sigma_k \sigma_j P^k}{\tilde{E} + m} \tilde{u}_+(\vec{P}') \right) d^3 \vec{P} d^3 \vec{P}' \\ &= \frac{1}{(2\pi\hbar L_p)^3} \frac{1}{2m} \int e^{\frac{i}{\hbar} (\vec{P} - \vec{P}') \vec{r}} e^{\frac{i}{\hbar} [(\vec{P} - \vec{P}') \vec{r} - c(\sqrt{M^2 c^2 + \vec{P}^2} - \sqrt{M^2 c^2 + \vec{P}'^2})] t} \\ &\quad i \varepsilon_{jkl} (P'^k + P^k) \tilde{u}_+^\dagger(\vec{P}) \sigma_l \tilde{u}_+(\vec{P}') d^3 \vec{P} d^3 \vec{P}'. \end{aligned} \quad (74)$$

From (68) with (71) and (74), we obtain the matrix element of the scattering potential  $V$ . With the algebraic formula

$$\delta(\vec{P} - \vec{P}') = \frac{1}{(2\pi\hbar)^3} \int e^{\frac{i}{\hbar}(\vec{P} - \vec{P}')\vec{r}} d^3\vec{r} \quad (75)$$

for a momentum conservation  $\vec{P} - \vec{P}'$ , when, in fact, no scattering exists, by integration in the infinite space we obtain a null matrix element:

$$\begin{aligned} V_{i0} &= \int \psi_{ti}^\dagger(\vec{r}, t) V \psi_{t0}(\vec{r}, t) d^3\vec{r} \\ &= -i\hbar \int \left[ \frac{\partial}{\partial t} [\psi_{ti}^\dagger(\vec{r}, t) \gamma^0 \psi_{t0}(\vec{r}, t)] + c \frac{\partial}{\partial x^j} \psi_{ti}^\dagger(\vec{r}, t) \gamma^j \psi_{t0}(\vec{r}, t) \right] d^3\vec{r} \\ &= \frac{1}{(2\pi\hbar L_p)^3} \frac{1}{2m} \int d^3\vec{P} d^3\vec{P}' e^{\frac{i}{\hbar}(\vec{P} - \vec{P}')\vec{r}} e^{\frac{i}{\hbar}[(\vec{P} - \vec{P}')\dot{\vec{r}} - c(\sqrt{M^2 c^2 + \vec{P}^2} - \sqrt{M^2 c^2 + \vec{P}'^2})]t} \left\{ (\tilde{E} + m) \right. \\ &\quad \left[ (\vec{P} - \vec{P}')\dot{\vec{r}} - c(\sqrt{M^2 c^2 + \vec{P}^2} - \sqrt{M^2 c^2 + \vec{P}'^2}) \right] \left[ 1 - \frac{\vec{P}\vec{P}' + i\vec{\sigma}(\vec{P} \times \vec{P}')}{(\tilde{E} + m)^2} \right] \tilde{u}^\dagger(\vec{P}) \tilde{u}(\vec{P}') \\ &\quad \left. + i\varepsilon_{jkl} c(P^j - P'^j)(P'^k + P^k) \tilde{u}^\dagger(\vec{P}) \sigma_l \tilde{u}(\vec{P}') \right\} d^3\vec{r} \\ &= 0. \end{aligned} \quad (76)$$

A non-null matrix element is obtained with a momentum variation in the collisional process. In this case, the two moments  $\vec{P}$  and  $\vec{P}'$  are integrated around their mean values  $\vec{P}_0$  and  $\vec{P}_i$ . We obtain this matrix element as an expression

$$\begin{aligned} V_{i0} &= \frac{1}{(2\pi\hbar L_p)^3} \int d^3\vec{P} d^3\vec{P}' e^{\frac{i}{\hbar}(\vec{P} - \vec{P}_0) - (\vec{P}' - \vec{P}_i)\vec{r}} e^{\frac{i}{\hbar}[(\vec{P} - \vec{P}')\dot{\vec{r}} - c(\sqrt{M^2 c^2 + \vec{P}^2} - \sqrt{M^2 c^2 + \vec{P}'^2})]t} f(\vec{P}, \vec{P}') d^3\vec{r} \\ &= \frac{1}{L_p^3} \int d^3\vec{P} d^3\vec{P}' e^{\frac{i}{\hbar}[(\vec{P} - \vec{P}')\dot{\vec{r}} - c(\sqrt{M^2 c^2 + \vec{P}^2} - \sqrt{M^2 c^2 + \vec{P}'^2})]t} f(\vec{P}, \vec{P}') \delta[(\vec{P} - \vec{P}_0) - (\vec{P}' - \vec{P}_i)] \\ &= \frac{1}{L_p^3} \int d^3\vec{P} e^{\frac{i}{\hbar}[\vec{P}_0\dot{\vec{r}} - \vec{P}_i\dot{\vec{r}} - c(\sqrt{M^2 c^2 + \vec{P}_0^2} - \sqrt{M^2 c^2 + \vec{P}_i^2})]t} f(\vec{P}_0, \vec{P}_i) \\ &= e^{\frac{i}{\hbar}[\vec{P}_0\dot{\vec{r}} - \vec{P}_i\dot{\vec{r}} - c(\sqrt{M^2 c^2 + \vec{P}_0^2} - \sqrt{M^2 c^2 + \vec{P}_i^2})]t} f(\vec{P}_0, \vec{P}_i), \end{aligned} \quad (77)$$

depending on the transition function

$$\begin{aligned} f(\vec{P}_0, \vec{P}_i) &= \frac{1}{2m} \left\{ (\tilde{E} + m) \left[ \vec{P}_0\dot{\vec{r}} - \vec{P}_i\dot{\vec{r}} - c(\sqrt{M^2 c^2 + \vec{P}_0^2} - \sqrt{M^2 c^2 + \vec{P}_i^2}) \right] \right. \\ &\quad \left[ 1 - \frac{\vec{P}_0\vec{P}_i + i\vec{\sigma}(\vec{P}_0 \times \vec{P}_i)}{(\tilde{E} + m)^2} \right] \tilde{u}^\dagger(\vec{P}_0) \tilde{u}(\vec{P}_i) + i\varepsilon_{jkl} c(P_0^j - P_i^j)(P_0^k + P_i^k) \tilde{u}^\dagger(\vec{P}_0) \sigma_l \tilde{u}(\vec{P}_i) \left. \right\}. \end{aligned} \quad (78)$$

Thus, from Fermi's golden rule (58) with these expressions, we obtain the scattering rate in a channel  $(i)$  defined by the final momenta  $\vec{P}'_i$ , of a particle with an initial momentum  $\vec{P}_0$ ,

$$\Gamma_{(i)} = \frac{2\pi}{\hbar^2} f(\vec{P}_0, \vec{P}'_i) f^*(\vec{P}_0, \vec{P}'_i) g(\omega_i) \quad (79)$$

In a collisional process with the spin conservation,

$$\tilde{u}(\vec{P}_0) \rightarrow \tilde{u}(\vec{P}'_i), \quad \tilde{u}(\vec{P}_0) = \tilde{u}(\vec{P}'_i) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad (80)$$

with the spin factor of the first term of the transition function (78),

$$\tilde{u}_+^\dagger(\vec{P}_0) \tilde{u}_+(\vec{P}'_i) = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1 + 0 = 1,$$

and the spin factor of the second term,

$$\begin{aligned} \varepsilon_{jkl} \tilde{u}^\dagger(\vec{P}_0) \sigma_l \tilde{u}(\vec{P}'_i) &= \tilde{u}^\dagger(\vec{P}_0) (\varepsilon_{jk1} \sigma_1 + \varepsilon_{jk2} \sigma_2 + \varepsilon_{jk3} \sigma_3) \tilde{u}_+(\vec{P}'_i) \\ &= \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} \varepsilon_{jk3} & \varepsilon_{jk1} - i\varepsilon_{jk2} \\ \varepsilon_{jk1} + i\varepsilon_{jk2} & -\varepsilon_{jk3} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} \varepsilon_{jk3} \\ \varepsilon_{jk1} + i\varepsilon_{jk2} \end{pmatrix} = \varepsilon_{jk3} \end{aligned}$$

the transition function (78) is

$$\begin{aligned} f(\vec{P}_0, \vec{P}'_i) &= \frac{1}{2m} \left\{ (\tilde{E} + m) \left[ \vec{P}_0 \cdot \vec{P}'_i - \vec{P}'_i \cdot \vec{P}_0 - c \left( \sqrt{M^2 c^2 + \vec{P}_0^2} - \sqrt{M^2 c^2 + \vec{P}'_i^2} \right) \right] \right. \\ &\quad \left. \left[ 1 - \frac{\vec{P}_0 \cdot \vec{P}'_i + i\vec{\sigma}(\vec{P}_0 \times \vec{P}'_i)}{(\tilde{E} + m)^2} \right] + i\varepsilon_{jk3} c (P_0^j - P_i'^j) (P_0^k + P_i'^k) \right\}. \end{aligned} \quad (81)$$

In a collisional process with the spin inversion,

$$\tilde{u}(\vec{P}_0) \rightarrow \tilde{u}(\vec{P}'_i), \quad \tilde{u}(\vec{P}_0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \tilde{u}(\vec{P}'_i) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad (82)$$

with the spin factor of the first term of the transition function (78)

$$\tilde{u}^\dagger(\vec{P}_0) \tilde{u}(\vec{P}'_i) = 0,$$

and the spin factor of the second term

$$\begin{aligned} i\varepsilon_{jkl} \tilde{u}^\dagger(\vec{P}_0) \sigma_l \tilde{u}(\vec{P}'_i) &= i\tilde{u}^\dagger(\vec{P}_0) (\varepsilon_{jk1} \sigma_1 + \varepsilon_{jk2} \sigma_2 + \varepsilon_{jk3} \sigma_3) \tilde{u}(\vec{P}'_i) \\ &= i \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} \varepsilon_{jk3} & \varepsilon_{jk1} - i\varepsilon_{jk2} \\ \varepsilon_{jk1} + i\varepsilon_{jk2} & -\varepsilon_{jk3} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = i \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} \varepsilon_{jk1} - i\varepsilon_{jk2} \\ -\varepsilon_{jk3} \end{pmatrix} \\ &= i(\varepsilon_{jk1} - i\varepsilon_{jk2}) = \varepsilon_{jk2} + i\varepsilon_{jk1}, \end{aligned}$$

the transition function (78) is

$$f(\vec{P}_0, \vec{P}'_i) = \frac{c}{2m} (\varepsilon_{jk2} + i\varepsilon_{jk1}) (P_0^j - P_i'^j) (P_0^k + P_i'^k). \quad (83)$$

## CONCLUSIONS

Here we discussed fundamental problems of our new quantum theory, of a distribution of matter propagating according to the general theory of relativity, and quantized as a mass of this distribution equal to the mass as a dynamic characteristic of its propagation. Unlike the conventional quantum mechanics, where the Hamiltonian of a quantum particle determines the partial time variation of the wave function, according to our theory the Hamiltonian determines the total time variation of the wave function. Besides the partial time derivative, the total time derivative includes terms as products of the spatial derivatives with the velocities, leading to a partial time derivative determined by the particle Lagrangian. This means a propagation wave function with a time dependent phase proportional to the Lagrangian, as the coordinate dependent phase is proportional to the momentum. It is remarkable that the velocities of these waves are equal to the distribution velocity, determined by these waves as the Fourier components. Essentially, a wave function describes a propagation in time of a Fourier series expansion in space of a distribution function, which, in this way, can be interpreted as representing a distribution of matter. It is remarkable that, in the famous representation of the interference of the waves determined by two slits, the dots on the screen do not represent particles, reaching different points with probabilities given by the wave function, but interactions of this wave functions with the localized molecular systems of the screen. In this representation, a particle does not pass only through one slit, taking an interferential trajectory by a mysterious knowledge about the existence of the other slit, as it is considered according to the conventional quantum mechanics. According to our theory, the particle is a distribution of matter passing through both slits, but can be understood/detected only as a quantum of matter with a mass equal to the mass in the time dependent phase, as a characteristic of the relativistic dynamics of the continuous matter composing this particle. The interaction of a quantum particle with a field is described by a time variation of the time-space interval, as the essential characteristic of the general theory of relativity, standing at the basis of our theory. If we consider a punctual particle as in the conventional quantum mechanics, the action of a field is described by a variation of the time-space interval between two arbitrary points of its trajectory. We obtained that the velocity induced to a particle at rest, by an arbitrary field, is equal to the light velocity, which means that, according to the theory of relativity, a punctual particle has no mass. This is a perfectly understandable result, because such a particle has no volume, no volume – no mass. In this way, we avoid the strange explanation of the particle mass as a coupling to a field of bosons, Higgs bosons. According to our theory, the mass of a particle is a characteristic of the relativistic dynamics of the continuous matter composing this particle, which is equal to the mass as an integral of the matter density.

We defined the density operator, and the density function as the diagonal elements of this operator in the space of the coordinate eigenstates. For the density operator, we obtained a dynamic equation depending on the particle Hamiltonian and velocity, and for the density function an equation of conservation, depending on a quantum matter flow density with a drift term proportional to the velocity, and a diffusion term proportional to the gradient of the wave function. Fermi's golden rule, for a scattering or a tunneling process, as a decay from the initial state of an incident particle to the final states of this particle, is reobtained from the dynamic equation of the density matrix with a perturbing potential. For a classical case, the transition matrix element is obtained as a particle matter flow, through the surface defining the states of a scattering center, or of the potential barrier in a tunneling process. For a relativistic case, we

obtain a scattering/tunneling rate depending on spinors, the relativistic energy, and the two momenta of the initial and the final states. We obtained these rates for the spin conservation, or inversion.

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