

On the Existence and Uniqueness of Solution for the Fractional Fredholm Integro-Differential Equation Through the Fractional Variational Orthogonal Collocation Method with Mamadu-Njoseh Orthogonal Basis Functions

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ABSTRACT

In this article, we present a detailed analysis on the existence and uniqueness of solution for the fractional Fredholm integro-differential equation (FFIDE) via the fractional variational orthogonal collocation method (FVOCM) with Mamadu-Njoseh orthogonal basis functions. Relevant theorems were formulated and proven to show that there exist a solution for the problem FFIDE through the FVOCM, and the solution is shown to be unique.

Keywords: Mamadu-Njoseh polynomials, Caputo derivative, Integro-differential equation, Basis functions, orthogonal polynomials, existence and uniqueness of solution.

INTRODUCTION

Many mathematical models of science and technology are described by fractional integro-differential equations. The approximate computations of these models have been the central issue of engineering and scientific computing. Evidently, with the development of supercomputers, many numerical or iterative schemes have been developed to solve these problems at ease such as found in the researches [1-10].

Now, a functional equation of the form [7, 9]

$$D^\alpha u(t) = f(t) + z(t), \quad t \in [0,1], \quad (n-1) < \alpha \leq n, n \in \mathbb{N} \quad (1.1)$$

with
$$z(t) = \int_a^b k(x,t)u(t)dt \quad (1.2)$$

$$u_i(t) = A_i, \quad i = 1,2,3, \dots, \quad (1.3)$$

is called the fractional order Fredholm integro-differential equations. Here, $u(t)$ is the unknown function, $D^\alpha u(t)$ is the Caputo order fractional derivative of $u(t)$, $k(x,t)$ is the kernel, and $f(t)$ is the source term. Further related literatures on the subject area can be found in the researches [11-12].

In this article, we present a detailed analysis on the existence and uniqueness of solution for the fractional Fredholm integro-differential equation (1.1) via the FVOCM with Mamadu-Njoseh orthogonal basis functions.

Now, we rewrite (1.1) - (1.3) in the form

$$\frac{d^\alpha u(t)}{dt^\alpha} = f(t) + \int_a^b k(x, t)u(t)dt, \quad (1.4)$$

$$u_i(t) = A_i, \quad i = 1, 2, 3, \dots,$$

and $t \in \Omega \subseteq \mathbb{R}^n$, where Ω is bounded and closed.

Define the vector function $u(t) = (u_1(t), u_2(t), \dots, u_n(t))$ such that it is continuous and analytic on

$$t \in \mathbb{R}^1 \times \Omega \times \Omega_1 = \alpha \times \Omega \times \Omega_1, \quad \alpha \in (-\infty, \infty), \Omega_1 \subseteq \mathbb{R}^n. \quad (1.5)$$

Suppose that $u(t)$ is Lipchitz continuous, i.e., satisfying the following inequalities:

$$\|u(t)\| \leq K, \quad (1.6)$$

$$\|u(t_1) - u(t_2)\| \leq K_1 \|t_1 - t_2\|, \quad (1.7)$$

for all $t \in \mathbb{R}^1$, $t_1 \in \Omega$, $t_2 \in \Omega_1$, and $K, K_1 > 0$.

The Fractional Variational Orthogonal Collocation Method (FVOCM)

This is a merger numerical approach for seeking the approximate solution of fractional Fredholm integro-differential equation with the aid of Mamadu-Njoseh orthogonal polynomials as basic functions.

The method is a mixture of the standard variational iteration method and the orthogonal collocation method. The method is simple, accurate and reliable with no weak assumptions or transformations. A brief review of the method with reference to the present work is presented in the following steps:

Step 1: Construct a correction functional for governing equation (1.1):

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda(s) \left[D^\alpha u_n(s) - f(s) - \int_a^b k(x, s)u_n(s)ds \right] (ds)^\alpha, \quad n \geq 1, \quad (2.1)$$

where $\lambda(s)$ is the general lagrange multiplier given as [13]

$$\lambda(s) = \frac{1}{\Gamma(1+\alpha)},$$

Step 2: Define an approximate solution (initial approximation) of the form:

$$u_0(t) = \sum_{i=0}^N a_i \varphi_i^\alpha(t), \quad t \in [-1, 1], \quad (2.2)$$

where a_i 's are constant parameters to be computed, and $\varphi_i^\alpha(t)$ are the Mamadu-Njoseh orthogonal polynomials [14 -15] in the Caputo sense with respect to the weight function $w(t) = 1 + t^2$.

Step 3: Choose a value for N in (2.2) and collocate orthogonally at $\varphi_{N+1}^\alpha(t)$ and solve the resulting algebraic equations for the values of a_i 's.

Step 4: Substitute the estimated values of a_i 's into (2.2) to obtain the initial approximation.

Step 5: Implement the iterative scheme (2.1) subject to the initial approximation for $n \geq 1$.

MAIN RESULTS: EXISTENCE AND UNIQUENESS OF SOLUTION

Theorem 1

If the equation (1.4) is Lipchitz continuous i.e., satisfies the inequalities (1.6) and (1.7) with the solution $u = u(t_0)$, then the iterates of functions generated via the FVOCM is given as:

$$u_{n+1}(t) = u_n(t) + \frac{1}{\Gamma(1+\alpha)} \int_0^t \left[\frac{d^\alpha}{ds^\alpha} (u_n(s)) - f(s) - \int_a^b k(x, s) u_n(s) ds \right] (ds)^\alpha, \quad (3.1)$$

where

$$u_n(t) = \sum_{i=0}^N a_i \varphi_i^\alpha(t). \quad (3.2)$$

Equation (3.1) with

$$u_0(t_0) = u_0, \quad n = 0, 1, 2, \dots,$$

converges absolutely with respect to:

$$u_0 \in \mathbb{R}^1 \times \Omega - K_1 \quad (3.3)$$

to the function $U_\alpha(t_0)$ defined in (3.3), and thus satisfying the equations:

$$u(t) = u_n(t) + \frac{1}{\Gamma(1+\alpha)} \int_0^t \left[\frac{d^\alpha}{ds^\alpha} (u_n(s)) - f(s) - \int_a^b k(x, s) u_n(s) ds \right] (ds)^\alpha, \quad (3.4)$$

provided that

$$\|u_\alpha - u_0\| \leq K_1.$$

Proof: Suppose that $\frac{d^\alpha}{dt^\alpha} u_n(t)$, $k(x, t)$ and $f(t)$ are continuous in t satisfying the following inequalities:

$$\left\| \frac{d^\alpha}{dt^\alpha} u_n(t) \right\| \leq G < \infty \quad (3.5)$$

$$\|f(t)\| \leq \gamma \quad (3.6)$$

$$\|u_0\| = \delta_0 \quad (3.7)$$

$$\left. \begin{aligned} \int_a^b \|k(x, t)\| dt &\leq M, \quad M > 0 \\ T_1 &= \sum_{i=1}^{\infty} M^i K_1^{i-1} \\ T_2 &= \sum_{i=1}^{\infty} i M^i K_1^{i-1} \end{aligned} \right\} \quad (3.8)$$

where $-\infty < t \leq T < \infty$, G, γ and δ_0 some are positive constants, and where T_1 and T_2 are convergent series.

Also, we define the non- empty sets:

$$\Omega_f = \Omega - K_1$$

$$\Omega_{1f} = \Omega_1 - K_1$$

where

$$K_1 = TG(\gamma\delta_0 + K), K_2 = G\delta_0 + G(\gamma\delta_0 + K), \|\cdot\| = \max_{0 \leq t \leq T}$$

And also

$$Z = TG(\gamma\delta_0 + K_1) < 1.$$

Let $u_1(t), u_2(t), \dots, u_n(t)$ be the iterates generated by the recurrence formula (3.1). Each of these iterates is well defined in the domain (1.5) and also continuous.

Now, when $n = 1$ using (3.1), we get,

$$\begin{aligned}\|u_1(t_0) - u_0(t_0)\| &= \left\| \frac{1}{\Gamma(1+\alpha)} \int_0^t \left[\frac{d^\alpha}{ds^\alpha} (u_0(s) - f(s) - \int_a^b k(x,s)u_0(s)ds) \right] (ds)^\alpha \right\| \\ &\leq \tau \int_0^t \left[\left\| \frac{d^\alpha}{ds^\alpha} (u_0(s)) \right\| - \|f(s)\| - \int_a^b \|k(x,s)\| \|u_0(s)\| ds \right] (ds)^\alpha \\ &\leq \tau T(G - \gamma - M\delta_0) = K_1, \quad \tau = \left\| \frac{1}{\Gamma(1+\alpha)} \right\|.\end{aligned}$$

Therefore,

$$\|u_1(t_0) - u_0(t_0)\| \leq K_1.$$

Now, suppose that $u_q(t_0) \in \Omega$, $\forall t \in \mathbb{R}^1$, $t_0 \in \Omega$, $q \in \mathbb{Z}^+$,

when $n = q + 1$, we have that

$$\|u_{q+1}(t_0) - u_0(t_0)\| \leq K_1, \quad u_{q+1}(t_0) \in \Omega, \quad \forall t \in \mathbb{R}^1, \quad t_0 \in \Omega, \quad q \in \mathbb{Z}^+.$$

Thus, we have by mathematical induction,

$$\|u_n(t_0) - u_0(t_0)\| \leq K_1, \quad \text{for } n = 0, 1, 2, \dots, \quad (3.9)$$

Similarly, we have:

$$\begin{aligned}\|u_1(t_0)\| &= \left\| u_0(t_0) + \frac{1}{\Gamma(1+\alpha)} \int_0^t \left[\frac{d^\alpha}{ds^\alpha} (u_0(s)) - f(s) - \int_a^b k(x,s)u_0(s)ds \right] (ds)^\alpha \right\| \\ &\leq \|u_0(t_0)\| + \tau \int_0^t \left[\left\| \frac{d^\alpha}{ds^\alpha} (u_0(s)) \right\| - \|f(s)\| - \int_a^b \|k(x,s)\| \|u_0(s)\| ds \right] (ds)^\alpha \\ &\leq \delta_0 + \tau TG(G - \gamma - M\delta_0) = K_2,\end{aligned}$$

and hence

$$\|u_1(t_0)\| \leq K_2.$$

Thus by mathematical induction, we have

$$\|u_n(t_0)\| \leq K_2, \quad \text{for all } n = 0, 1, 2, \dots \quad (3.10)$$

Next, we show that the iterates of function (3.1) are uniformly convergent on $u_0 \in \mathbb{R}^1 \times \Omega_f$.

Let $n = 1$ in (3.1), we have that

$$\|u_2(t_0) - u_1(t_0)\| \leq \tau \int_0^t \left[\left\| \frac{d^\alpha}{ds^\alpha} (u_2(s) - u_1(s)) \right\| + \|u_1(s) - u_0(s)\| \right] (ds)^\alpha \quad (3.11)$$

Using (3.9) and (3.10), we get

$$\|u_2(t_0) - u_1(t_0)\| \leq \tau TGK_1.$$

Also, when $n = q$ in (3.1), we get,

$$\|u_q(t_0) - u_0(t_0)\| \leq \tau TGK_1^{p-1}.$$

Now, suppose that the inequality

$$\|u_{q+1}(t_0) - u_q(t_0)\| \leq TQK_1^{p-1} \quad (3.12)$$

holds for $n = q$, then it can be shown that the inequality

$$\|u_{q+2}(t_0) - u_{q+1}(t_0)\| \leq TQK_1 \quad (3.13)$$

holds for $n = q + 1$.

Thus,

$$\|u_{q+2}(t_0) - u_{q+1}(t_0)\| \leq \tau \int_0^t \left[\left\| \frac{d^\alpha}{ds^\alpha} (u_{q+1}(s) - u_q(s)) \right\| + M\|u_{q+1}(s) - u_q(s)\| \right] (ds)^\alpha$$

By (3.12), we get

$$\begin{aligned} & \|u_{q+1}(t_0) - u_q(t_0)\| \\ & \leq \tau \int_0^t G \|u_{q+1}(s) - u_q(s)\| + K_1 \left(\sum_{i=1}^{\infty} i M^i K_1^{i-1} \right) \|u_{q+1}(s) - u_q(s)\| (ds)^\alpha. \end{aligned}$$

Hence,

$$\|u_{q+2}(t_0) - u_q(t_0)\| \leq \tau TG(\gamma TGK_1^{p-1} + KTGK_1 K_1^{p-1})$$

By mathematical induction, the following inequality holds:

$$\|u_{n+1}(t_0) - u_n(t_0)\| \leq \tau TGK, K = TGK_1^{p-1}(\gamma + K_1),, \quad n = 1, 2, \dots \quad (3.14)$$

Hence, from (3.14), the following inequalities holds for $r > 1$

$$\|u_{n+r}(t_0) - u_r(t_0)\| = \sum_{j=0}^{\infty} \|u_{n+1+j}(t_0) - u_{n+j}(t_0)\|, \quad \forall n.$$

Hence,

$$\begin{aligned} \|u_{n+r}(t_0) - u_n(t_0)\| & \leq \sum_{j=0}^{\infty} \|u_{n+1+j}(t_0) - u_{n+j}(t_0)\| \\ & \leq \sum_{j=0}^{\infty} TGK_1 K^{n-1+j} \\ & \leq TGK_1 K^{n-1} \sum_{j=0}^{\infty} K^j \\ & \leq TGK_1 (1 - K)^{-1} K^{n-1}. \end{aligned}$$

Thus,

$$\|u_{n+1}(t_0) - u_n(t_0)\| \leq TGK_1 (1 - K)^{-1} K^{n-1}. \quad (3.15)$$

Using (3.15) for $n = 1, 2, 3, \dots$, we obtain

$$\lim_{n \rightarrow \infty} K^{n-1} = 0 \Rightarrow \lim_{n \rightarrow \infty} u_n(t_0) = u_\infty(t_0) \quad (3.16)$$

Thus, the equations (3.15) and (3.16) show that the iterates of the function (3.1) converges uniformly on the domain $u_0 \in \mathbb{R}^1 \times \Omega_f$ as $n \rightarrow \infty$. Hence, the limiting function $u_\infty(t_0)$ exist as the solution of the fractional Fredholm integro-differential equations (1.4).

Theorem 2

Supposing all the conditions and assumptions of theorem 1 are true, then the function $u(t_0)$ is a unique solution of (1.4).

Proof: Assume that

$$u_{n+1}(t) = u_n(t) + \frac{1}{\Gamma(1+\alpha)} \int_0^t \left[\frac{d^\alpha}{ds^\alpha} (u_n(s)) - f(s) - \int_a^b k(x, s) u_n(s) ds \right] (ds)^\alpha$$

and

$$u_{m+1}(t) = u_m(t) + \frac{1}{\Gamma(1+\alpha)} \int_0^t \left[\frac{d^\alpha}{ds^\alpha} (u_m(s)) - f(s) - \int_a^b k(x, s) u_m(s) ds \right] (ds)^\alpha,$$

are solutions of (1.4).

Now,

$$\begin{aligned} & \|u_{n+1}(t_0) - u_{m+1}(t_0)\| \\ & = \left\| u_n(t_0) + \frac{1}{\Gamma(1+\alpha)} \int_0^t \left[\frac{d^\alpha}{ds^\alpha} (u_n(s)) - f(s) - \int_a^b k(x, s) u_n(s) ds \right] (ds)^\alpha - u_m(t_0) \right. \\ & \quad \left. - \frac{1}{\Gamma(1+\alpha)} \int_0^t \left[\frac{d^\alpha}{ds^\alpha} (u_m(s)) - f(s) - \int_a^b k(x, s) u_m(s) ds \right] (ds)^\alpha \right\| \end{aligned}$$

$$\|u_{n+1}(t_0) - u_{m+1}(t_0)\| \leq \|u_n(t_0) - u_m(t_0)\| + \tau \int_0^t \left\| \frac{d^\alpha}{ds^\alpha} (u_n(s) - u_m(s)) \right\| + \int_a^b \|k(x, \tau)\| \|u_m(s) - u_n(s)\| ds \Big] (ds)^\alpha$$

By using all conditions and assumptions of Theorem 1, we get

$$\|u_n(t_0) - u_m(t_0)\| \leq \tau TG + K_1 \left(\sum_{i=1}^{\infty} i M^i K_1^{i-1} \right) \|u_n(t_0) - u_m(t_0)\| + K_1 \quad (3.17)$$

Also, we find:

$$\begin{aligned} \|u_n(t_0) - u_m(t_0)\| &\leq G \|u_n(t_0) - u_m(t_0)\| + K_1 \|u_n(t_0) - u_m(t_0)\| \\ &\leq (G + K_1) \|u_n(t_0) - u_m(t_0)\| \end{aligned} \quad (3.18)$$

From the inequalities (3.17) and (3.18), we obtain

$$\|u_n(t_0) - u_m(t_0)\| \leq (TG + K_1)^2 \|u_n(t_0) - u_m(t_0)\|$$

Thus,

$$\|u_n(t_0) - u_m(t_0)\| \leq DK,$$

Where

$$D = (TG + K_1)^2 \|u_n(t_0) - u_m(t_0)\|.$$

By mathematical induction, we have that

$$\|u_n(t_0) - u_m(t_0)\| \leq DK^n \quad (3.19)$$

Therefore, using (3.19) and (3.16) for $n \rightarrow \infty$, we have that $K^n \rightarrow 0$.

Thus,

$$u_n(t_0) = u_m(t_0)$$

Thus, $u(t_0)$ is a unique solution of (1.4) $\forall t \in \mathbb{R}^1, t_0 \in \Omega_f$.

For example, the fractional integro-differential equation

$$D^{1/2}u(x) = \frac{(3/8)x^{3/2} - 2x^{1/2}}{\sqrt{\pi}} + \int_0^1 xtu(t)dt, \quad x \geq 0, t \leq 1,$$

with exact solution $u(x) = x(x-1)$, can be solved using the FVOCM through the steps as follows:

1. Correction functional for the problem is given as

$$u_{n+1}(x) = u_n(x) + \frac{1}{\Gamma(\alpha+1)} \int_0^t \left(D^{1/2}u_n(s) - \frac{(3/8)s^{3/2} - 2s^{1/2}}{\sqrt{\pi}} - \int_0^1 xtu_n(t)dt \right) (ds)^\alpha, n \geq 1.$$

2. Let $N = 3$ such that the initial approximation is

$$u_0(x) = \sum_{i=0}^N a_i \varphi_i^\alpha(x), \quad x \in [0,1].$$

where $\varphi_i^\alpha(x)$ is the shifted Mamadu-Njoseh orthogonal polynomials in the Caputo sense obtained through the transformation

$$x = \frac{1}{2}(t+1), \quad dx = \frac{1}{2}dt.$$

3. Collocating orthogonally at $\varphi_4^\alpha(x)$, and then solving the resulting algebraic equations to get $a_i = 0, i = 0(1)3$.

4. Implementing the iterative scheme subject to the initial approximation for $n \geq 1$, we obtain the following iterates with the aid of MAPLE 18;

$$u_1 := -\frac{4}{15} \frac{x^{3/2} (4x-5)}{\sqrt{\pi} \Gamma(\alpha+1)}$$

$$u_2 := -\frac{4}{15} \frac{x^{3/2} (4x-5)}{\sqrt{\pi} \Gamma(\alpha+1)} + \frac{1}{1890} \frac{1}{\Gamma(\alpha+1)^2 \sqrt{\pi}} (224x^{9/2} - 360x^{7/2} - 2016x^{5/2} \Gamma(0 + 1) + 2520x^{3/2} \Gamma(\alpha+1) - 504x^3 \sqrt{\pi} + 945x^2 \sqrt{\pi})$$

$$u_3 := -\frac{4}{15} \frac{x^{3/2} (4x-5)}{\sqrt{\pi} \Gamma(\alpha+1)} + \frac{1}{1890} \frac{1}{\Gamma(\alpha+1)^2 \sqrt{\pi}} (224x^{9/2} - 360x^{7/2} - 2016x^{5/2} \Gamma(0 + 1) + 2520x^{3/2} \Gamma(\alpha+1) - 504x^3 \sqrt{\pi} + 945x^2 \sqrt{\pi})$$

$$- \frac{1}{21621600} \frac{1}{\Gamma(\alpha+1)^3 \sqrt{\pi}} (197120x^{13/2} - 374400x^{11/2} - 5125120x^{9/2} \Gamma(\alpha+1) + 8236800x^{7/2} \Gamma(\alpha+1) + 5271552x^{7/2} + 23063040x^{5/2} \Gamma(\alpha+1)^2 - 1207206\sqrt{\pi}x^5 - 11531520x^{5/2} - 28828800x^{3/2} \Gamma(\alpha+1)^2 + 2477475x^4 \sqrt{\pi})$$

$$\vdots$$

The Table below show the maximum errors obtained for each iterate above for $\alpha = 0.5$ compared with the standard least square method (SLSM) and perturbed least square method (PLSM).

Table 1. Comparison of Maximum Errors at $\alpha = 0.5$

x	FVOCM, u_3	SLSM [6]	PLSM [6]
0.1	1.0000×10^{-11}	1.6795×10^{-5}	9.1044×10^{-5}
0.5	5.0001×10^{-11}	1.5190×10^{-5}	7.0128×10^{-5}
1.0	1.0000×10^{-11}	3.5524×10^{-5}	1.4367×10^{-4}

DISCUSSION OF RESULTS

There exist several numerical schemes for solving fractional Fredholm integro-differential equations. Here, the proposed method has been compared with the SLSM and PLSM as available in the literature [6]. From Table 1, it is obvious that the proposed method attained a maximum error of order 10–11, while the SLSM and PLSM attained maximum errors of order 10–5 and 10–4 respectively. Thus, it is obvious that the proposed is a far better solver than the SLSM and PLSM as it converges rapidly to the exact solution. All computation proceedings were performed with MAPLE 18 software.

CONCLUSION

We have successfully carried out a detailed analysis on the existence and uniqueness of solution for the fractional Fredholm integro - differential equation (1.4) via FVOCM with Mamadu-Njoseh orthogonal basis functions. Our findings have shown that there exists a solution for the fractional Fredholm integro- differential equation through FVOCM with Mamadu-Njoseh orthogonal basis functions and that the solution is unique.

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