

Property of Tensor Satisfying Binary Law 4

Koji Ichidayama

Okayama 716-0002, Japan

ABSTRACT

I have already reported "Property of Tensor Satisfying Binary Law 3". This article improved a proof part in article "Property of Tensor Satisfying Binary Law 3". The Proposition that improvement was carried out is Proposition4-8, Proposition10-15 in article "Property of Tensor Satisfying Binary Law 3". The Proposition of article "Property of Tensor Satisfying Binary Law 3" will become stronger by this improvement.

Keywords: Tensor; Covariant Derivative.

INTRODUCTION

I have already reported "Property of Tensor Satisfying Binary Law 3".[2] This article improved a proof part in article "Property of Tensor Satisfying Binary Law 3".

The Proposition that improvement was carried out is Proposition4-8, Proposition10-15 in article "Property of Tensor Satisfying Binary Law 3".

The Proposition of article "Property of Tensor Satisfying Binary Law 3" will become stronger by this improvement.

In addition, I performed the explanation about this improvement in Discussion chapter. The improvement technique that I showed here was performed about other Proposition2-5 equally.

DEFINITION

Definition1 $\overline{x^\mu} \neq x^\mu, \overline{x^\nu} \neq x^\nu, \overline{x^\mu} = x^\nu, \overline{x^\nu} = x^\mu$ is established.[1] I named $\overline{x^\mu} \neq x^\mu, \overline{x^\nu} \neq x^\nu, \overline{x^\mu} = x^\nu, \overline{x^\nu} = x^\mu$ "Binary Law".[1]

Definition2 If $\overline{x^\mu} \neq x^\mu, \overline{x^\nu} \neq x^\nu, \overline{x^\mu} = x^\nu, \overline{x^\nu} = x^\mu$ is established, $x_\nu = x^\mu$ is established.[1]

Definition3 If $\overline{x^\mu} \neq x^\mu, \overline{x^\nu} \neq x^\nu, \overline{x^\mu} = x^\nu, \overline{x^\nu} = x^\mu$ is established, $x_\mu = x^\nu$ is established.[1]

Definition4 If $\overline{x^\mu} \neq x^\mu, \overline{x^\nu} \neq x^\nu, \overline{x^\mu} = x^\nu, \overline{x^\nu} = x^\mu$ is established, $x_\nu = -x_\mu$ is established.[1]

Definition5 If all coordinate systems $x^\mu, x^\nu, x^\sigma, x^\lambda, \dots$ satisfies $\overline{x^\mu} \neq x^\mu, \overline{x^\nu} \neq x^\nu, \overline{x^\mu} = x^\nu, \overline{x^\nu} = x^\mu$,
all coordinate systems $x^\mu, x^\nu, x^\sigma, x^\lambda, \dots$ shifts to only two of x^μ, x^ν . [1]

Definition6 $g_{\mu}^{\mu} = 1, g_{\nu}^{\mu} = 0: (\mu \neq \nu)$ is establishment.[3]

Definition7 $\frac{\partial^3 x^{\mu}}{\partial x^{\nu} \partial x^{\nu} \partial x^{\nu}} = M$ is established for $\frac{\partial^3 x^{\mu}}{\partial x^{\nu} \partial x^{\nu} \partial x^{\nu}}$.

Definition8 The first-order covariant derivative of the covariant vector satisfied

$$x_{\mu;\nu} = \frac{\partial x_{\mu}}{\partial x^{\nu}} - x_{\tau} \Gamma_{\mu\nu}^{\tau} = \frac{\partial x_{\mu}}{\partial x^{\nu}} - x_{\tau} \frac{1}{2} g^{\epsilon\tau} \left(\frac{\partial g_{\mu\epsilon}}{\partial x^{\nu}} + \frac{\partial g_{\nu\epsilon}}{\partial x^{\mu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\epsilon}} \right). [4]$$

Definition9 The first-order covariant derivative of the contravariant vector satisfied

$$x^{\mu}_{;\nu} = \frac{\partial x^{\mu}}{\partial x^{\nu}} + x^{\tau} \Gamma_{\tau\nu}^{\mu} = \frac{\partial x^{\mu}}{\partial x^{\nu}} + x^{\tau} \frac{1}{2} g^{\epsilon\mu} \left(\frac{\partial g_{\tau\epsilon}}{\partial x^{\nu}} + \frac{\partial g_{\nu\epsilon}}{\partial x^{\tau}} - \frac{\partial g_{\tau\nu}}{\partial x^{\epsilon}} \right). [4]$$

Definition10 The second-order covariant derivative of the covariant vector satisfied

$$\begin{aligned} x_{\mu;\nu;\sigma} &= \frac{\partial x_{\mu;\nu}}{\partial x^{\sigma}} - x_{\iota;\nu} \Gamma_{\mu\sigma}^{\iota} - x_{\mu;\iota} \Gamma_{\nu\sigma}^{\iota} \\ &= \frac{\partial}{\partial x^{\sigma}} \left(\frac{\partial x_{\mu}}{\partial x^{\nu}} - x_{\tau} \Gamma_{\mu\nu}^{\tau} \right) - \left(\frac{\partial x_{\iota}}{\partial x^{\nu}} - x_{\tau} \Gamma_{\iota\nu}^{\tau} \right) \Gamma_{\mu\sigma}^{\iota} - \left(\frac{\partial x_{\mu}}{\partial x^{\iota}} - x_{\tau} \Gamma_{\mu\iota}^{\tau} \right) \Gamma_{\nu\sigma}^{\iota} \\ &= \frac{\partial^2 x_{\mu}}{\partial x^{\nu} \partial x^{\sigma}} - \frac{\partial}{\partial x^{\sigma}} \left(x_{\tau} \frac{1}{2} g^{\epsilon\tau} \left(\frac{\partial g_{\mu\epsilon}}{\partial x^{\nu}} + \frac{\partial g_{\nu\epsilon}}{\partial x^{\mu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\epsilon}} \right) \right) \\ &\quad - \frac{\partial x_{\iota}}{\partial x^{\nu}} \frac{1}{2} g^{\epsilon\iota} \left(\frac{\partial g_{\mu\epsilon}}{\partial x^{\sigma}} + \frac{\partial g_{\sigma\epsilon}}{\partial x^{\mu}} - \frac{\partial g_{\mu\sigma}}{\partial x^{\epsilon}} \right) + x_{\tau} \frac{1}{2} g^{\epsilon\tau} \left(\frac{\partial g_{\iota\epsilon}}{\partial x^{\nu}} + \frac{\partial g_{\nu\epsilon}}{\partial x^{\iota}} - \frac{\partial g_{\iota\nu}}{\partial x^{\epsilon}} \right) \frac{1}{2} g^{\epsilon\iota} \left(\frac{\partial g_{\mu\epsilon}}{\partial x^{\sigma}} + \frac{\partial g_{\sigma\epsilon}}{\partial x^{\mu}} - \frac{\partial g_{\mu\sigma}}{\partial x^{\epsilon}} \right) \\ &\quad - \frac{\partial x_{\mu}}{\partial x^{\iota}} \frac{1}{2} g^{\epsilon\iota} \left(\frac{\partial g_{\nu\epsilon}}{\partial x^{\sigma}} + \frac{\partial g_{\sigma\epsilon}}{\partial x^{\nu}} - \frac{\partial g_{\nu\sigma}}{\partial x^{\epsilon}} \right) + x_{\tau} \frac{1}{2} g^{\epsilon\tau} \left(\frac{\partial g_{\mu\epsilon}}{\partial x^{\iota}} + \frac{\partial g_{\iota\epsilon}}{\partial x^{\mu}} - \frac{\partial g_{\mu\iota}}{\partial x^{\epsilon}} \right) \frac{1}{2} g^{\epsilon\iota} \left(\frac{\partial g_{\nu\epsilon}}{\partial x^{\sigma}} + \frac{\partial g_{\sigma\epsilon}}{\partial x^{\nu}} - \frac{\partial g_{\nu\sigma}}{\partial x^{\epsilon}} \right). [4] \end{aligned}$$

Definition11 The second-order covariant derivative of the contravariant vector satisfied

$$\begin{aligned} x^{\mu}_{;\nu;\sigma} &= \frac{\partial x^{\mu}_{;\nu}}{\partial x^{\sigma}} + x^{\iota}_{;\nu} \Gamma_{\iota\sigma}^{\mu} - x^{\mu}_{;\iota} \Gamma_{\nu\sigma}^{\iota} \\ &= \frac{\partial}{\partial x^{\sigma}} \left(\frac{\partial x^{\mu}}{\partial x^{\nu}} + x^{\tau} \Gamma_{\tau\nu}^{\mu} \right) + \left(\frac{\partial x^{\iota}}{\partial x^{\nu}} + x^{\tau} \Gamma_{\tau\nu}^{\iota} \right) \Gamma_{\iota\sigma}^{\mu} - \left(\frac{\partial x^{\mu}}{\partial x^{\iota}} + x^{\tau} \Gamma_{\tau\iota}^{\mu} \right) \Gamma_{\nu\sigma}^{\iota} \\ &= \frac{\partial^2 x^{\mu}}{\partial x^{\nu} \partial x^{\sigma}} + \frac{\partial}{\partial x^{\sigma}} \left(x^{\tau} \frac{1}{2} g^{\epsilon\mu} \left(\frac{\partial g_{\tau\epsilon}}{\partial x^{\nu}} + \frac{\partial g_{\nu\epsilon}}{\partial x^{\tau}} - \frac{\partial g_{\tau\nu}}{\partial x^{\epsilon}} \right) \right) \\ &\quad + \frac{\partial x^{\iota}}{\partial x^{\nu}} \frac{1}{2} g^{\epsilon\mu} \left(\frac{\partial g_{\iota\epsilon}}{\partial x^{\sigma}} + \frac{\partial g_{\sigma\epsilon}}{\partial x^{\iota}} - \frac{\partial g_{\iota\sigma}}{\partial x^{\epsilon}} \right) + x^{\tau} \frac{1}{2} g^{\epsilon\iota} \left(\frac{\partial g_{\tau\epsilon}}{\partial x^{\nu}} + \frac{\partial g_{\nu\epsilon}}{\partial x^{\tau}} - \frac{\partial g_{\tau\nu}}{\partial x^{\epsilon}} \right) \frac{1}{2} g^{\epsilon\mu} \left(\frac{\partial g_{\iota\epsilon}}{\partial x^{\sigma}} + \frac{\partial g_{\sigma\epsilon}}{\partial x^{\iota}} - \frac{\partial g_{\iota\sigma}}{\partial x^{\epsilon}} \right) \\ &\quad - \frac{\partial x^{\mu}}{\partial x^{\iota}} \frac{1}{2} g^{\epsilon\iota} \left(\frac{\partial g_{\nu\epsilon}}{\partial x^{\sigma}} + \frac{\partial g_{\sigma\epsilon}}{\partial x^{\nu}} - \frac{\partial g_{\nu\sigma}}{\partial x^{\epsilon}} \right) - x^{\tau} \frac{1}{2} g^{\epsilon\mu} \left(\frac{\partial g_{\tau\epsilon}}{\partial x^{\iota}} + \frac{\partial g_{\iota\epsilon}}{\partial x^{\tau}} - \frac{\partial g_{\tau\iota}}{\partial x^{\epsilon}} \right) \frac{1}{2} g^{\epsilon\iota} \left(\frac{\partial g_{\nu\epsilon}}{\partial x^{\sigma}} + \frac{\partial g_{\sigma\epsilon}}{\partial x^{\nu}} - \frac{\partial g_{\nu\sigma}}{\partial x^{\epsilon}} \right). [4] \end{aligned}$$

Definition12 The third-order covariant derivative of the contravariant vector satisfied

$$\begin{aligned} x^{\mu}_{;\nu;\sigma;\lambda} &= \frac{\partial x^{\mu}_{;\nu;\sigma}}{\partial x^{\lambda}} + x^{\kappa}_{;\nu;\sigma} \Gamma_{\kappa\lambda}^{\mu} - x^{\mu}_{;\kappa;\sigma} \Gamma_{\nu\lambda}^{\kappa} - x^{\mu}_{;\nu;\kappa} \Gamma_{\sigma\lambda}^{\kappa} \\ &= \frac{\partial}{\partial x^{\lambda}} \left\{ \frac{\partial}{\partial x^{\sigma}} \left(\frac{\partial x^{\mu}}{\partial x^{\nu}} + x^{\tau} \Gamma_{\tau\nu}^{\mu} \right) + \left(\frac{\partial x^{\iota}}{\partial x^{\nu}} + x^{\tau} \Gamma_{\tau\nu}^{\iota} \right) \Gamma_{\iota\sigma}^{\mu} - \left(\frac{\partial x^{\mu}}{\partial x^{\iota}} + x^{\tau} \Gamma_{\tau\iota}^{\mu} \right) \Gamma_{\nu\sigma}^{\iota} \right\} \\ &\quad + \left\{ \frac{\partial}{\partial x^{\sigma}} \left(\frac{\partial x^{\kappa}}{\partial x^{\nu}} + x^{\tau} \Gamma_{\tau\nu}^{\kappa} \right) + \left(\frac{\partial x^{\iota}}{\partial x^{\nu}} + x^{\tau} \Gamma_{\tau\nu}^{\iota} \right) \Gamma_{\iota\sigma}^{\kappa} - \left(\frac{\partial x^{\kappa}}{\partial x^{\iota}} + x^{\tau} \Gamma_{\tau\iota}^{\kappa} \right) \Gamma_{\nu\sigma}^{\iota} \right\} \Gamma_{\kappa\lambda}^{\mu} \\ &\quad - \left\{ \frac{\partial}{\partial x^{\sigma}} \left(\frac{\partial x^{\mu}}{\partial x^{\kappa}} + x^{\tau} \Gamma_{\tau\kappa}^{\mu} \right) + \left(\frac{\partial x^{\iota}}{\partial x^{\kappa}} + x^{\tau} \Gamma_{\tau\kappa}^{\iota} \right) \Gamma_{\iota\sigma}^{\mu} - \left(\frac{\partial x^{\mu}}{\partial x^{\iota}} + x^{\tau} \Gamma_{\tau\iota}^{\mu} \right) \Gamma_{\kappa\sigma}^{\iota} \right\} \Gamma_{\nu\lambda}^{\kappa} \\ &\quad - \left\{ \frac{\partial}{\partial x^{\kappa}} \left(\frac{\partial x^{\mu}}{\partial x^{\nu}} + x^{\tau} \Gamma_{\tau\nu}^{\mu} \right) + \left(\frac{\partial x^{\iota}}{\partial x^{\nu}} + x^{\tau} \Gamma_{\tau\nu}^{\iota} \right) \Gamma_{\iota\sigma}^{\mu} - \left(\frac{\partial x^{\mu}}{\partial x^{\iota}} + x^{\tau} \Gamma_{\tau\iota}^{\mu} \right) \Gamma_{\nu\sigma}^{\iota} \right\} \Gamma_{\kappa\lambda}^{\mu} \\ &= \frac{\partial^3 x^{\mu}}{\partial x^{\nu} \partial x^{\sigma} \partial x^{\lambda}} + \frac{\partial^2}{\partial x^{\sigma} \partial x^{\lambda}} \left(x^{\tau} \frac{1}{2} g^{\epsilon\mu} \left(\frac{\partial g_{\tau\epsilon}}{\partial x^{\nu}} + \frac{\partial g_{\nu\epsilon}}{\partial x^{\tau}} - \frac{\partial g_{\tau\nu}}{\partial x^{\epsilon}} \right) \right) \\ &\quad + \frac{\partial}{\partial x^{\lambda}} \left(\frac{\partial x^{\iota}}{\partial x^{\nu}} \frac{1}{2} g^{\epsilon\mu} \left(\frac{\partial g_{\iota\epsilon}}{\partial x^{\sigma}} + \frac{\partial g_{\sigma\epsilon}}{\partial x^{\iota}} - \frac{\partial g_{\iota\sigma}}{\partial x^{\epsilon}} \right) \right) \\ &\quad + \frac{\partial}{\partial x^{\lambda}} \left(x^{\tau} \frac{1}{2} g^{\epsilon\iota} \left(\frac{\partial g_{\tau\epsilon}}{\partial x^{\nu}} + \frac{\partial g_{\nu\epsilon}}{\partial x^{\tau}} - \frac{\partial g_{\tau\nu}}{\partial x^{\epsilon}} \right) \frac{1}{2} g^{\epsilon\mu} \left(\frac{\partial g_{\iota\epsilon}}{\partial x^{\sigma}} + \frac{\partial g_{\sigma\epsilon}}{\partial x^{\iota}} - \frac{\partial g_{\iota\sigma}}{\partial x^{\epsilon}} \right) \right) \end{aligned}$$

Definition17 When the next conversion equation is established, $x_{\mu\nu}$ is covariant components of a tensor of the second rank.[4] $x_{\mu\nu} = \frac{\partial x^\sigma}{\partial x^\mu} \frac{\partial x^\lambda}{\partial x^\nu} x_{\sigma\lambda}$

Definition18 When the next conversion equation is established, x_v^μ is components of the mixed tensor of the second rank.[4] $x_v^\mu = \frac{\partial x^\mu}{\partial x^\sigma} \frac{\partial x^\lambda}{\partial x^\nu} x_\lambda^\sigma$

Definition19 When the next conversion equation is established, $x_{\mu\nu\sigma}$ is covariant components of a tensor of the third rank.[4] $x_{\mu\nu\sigma} = \frac{\partial x^\lambda}{\partial x^\mu} \frac{\partial x^\iota}{\partial x^\nu} \frac{\partial x^\epsilon}{\partial x^\sigma} x_{\lambda\iota\epsilon}$

Definition20 When the next conversion equation is established, $x_{\nu\sigma}^\mu$ is components of the mixed tensor of the third rank of the second rank covariant in the first rank contravariant.[4] $x_{\nu\sigma}^\mu = \frac{\partial x^\mu}{\partial x^\lambda} \frac{\partial x^\iota}{\partial x^\nu} \frac{\partial x^\epsilon}{\partial x^\sigma} x_{\iota\epsilon}^\lambda$

Definition21 When the next conversion equation is established, $x_{\nu\sigma\lambda}^\mu$ is components of the mixed tensor of the fourth rank of the third rank covariant in the first rank contravariant.[4] $x_{\nu\sigma\lambda}^\mu = \frac{\partial x^\mu}{\partial x^\iota} \frac{\partial x^\epsilon}{\partial x^\nu} \frac{\partial x^\alpha}{\partial x^\sigma} \frac{\partial x^\beta}{\partial x^\lambda} x_{\epsilon\alpha\beta}^\iota$

ABOUT COVARIANT DERIVATIVE FOR THE VECTOR IN TENSOR SATISFYING BINARY LAW

Proposition1 $x_{\mu;\nu} = \frac{\partial x_\mu}{\partial x^\nu}$, $x_\mu^{;\mu} = \frac{\partial x_\mu}{\partial x_\mu} - x_\nu \frac{1}{2} \frac{\partial g^{\mu\nu}}{\partial x^\mu}$ is established in tensor satisfying Binary Law.

Proof: If all coordinate systems satisfies Binary Law in $\frac{\partial g_{\mu\epsilon}}{\partial x^\nu} - \frac{\partial g_{\mu\nu}}{\partial x^\epsilon}$,

$$\frac{\partial g_{\mu\epsilon}}{\partial x^\nu} - \frac{\partial g_{\mu\nu}}{\partial x^\epsilon} = 0 \quad (1)$$

is established. I get

$$x_{\mu;\nu} = \frac{\partial x_\mu}{\partial x^\nu} - x_\tau \frac{1}{2} g^{\epsilon\tau} \left(\frac{\partial g_{\nu\epsilon}}{\partial x^\mu} \right) = \frac{\partial x_\mu}{\partial x^\nu} - x_\tau \frac{1}{2} \frac{\partial g_\nu^\tau}{\partial x^\mu} \quad (2)$$

from (1),Definision8. I get

$$x_{\mu;\nu} = \frac{\partial x_\mu}{\partial x^\nu} - x_\tau \frac{1}{2} \frac{\partial g_\nu^\tau}{\partial x^\mu} \quad (3)$$

as if Binary Law being satisfied for all index except the dummy index of (2). If (3) is a tensor equation, the dummy index of (3) can't make μ or ν . On the other hand, If all coordinate systems satisfies Binary Law, dummy index of (3) should be μ or ν in consideration of Definision5. I aim at the coexistence of these two demands. I rewrite (3) using Definision2 and get

$$x_\mu^{;\mu} = \frac{\partial x_\mu}{\partial x_\mu} - x_\tau \frac{1}{2} \left(\frac{\partial g^{\mu\tau}}{\partial x^\mu} \right) = \frac{\partial x_\mu}{\partial x_\mu} - x_\nu \frac{1}{2} \left(\frac{\partial g^{\mu\nu}}{\partial x^\mu} \right). \quad (4)$$

(4) satisfies two demands mentioned above together here. I rewrite (3) using Definision4 and get

$$-x_{\mu;\mu} = -\frac{\partial x_\mu}{\partial x^\mu} + x_\tau \frac{1}{2} \left(\frac{\partial g_\mu^\tau}{\partial x^\mu} \right) = -\frac{\partial x_\mu}{\partial x^\mu} + x_\nu \frac{1}{2} \left(\frac{\partial g_\mu^\nu}{\partial x^\mu} \right). \quad (5)$$

(5) satisfies two demands mentioned above together here. I get

$$-x_{\mu;\mu} = -\frac{\partial x_{\mu}}{\partial x^{\mu}} \quad (6)$$

in consideration of Definision6 for (5). Because the second term of the right side doesn't exist in (6),

$$x_{\mu;\nu} = \frac{\partial x_{\mu}}{\partial x^{\nu}} \quad (7)$$

can rewrite (6) using Definision4. In addition, $x_{\mu;\nu}$ can't rewrite x_{μ}^{μ} of (4) using Definision2 because the second term of the right side exists in (4). End Proof

Proposition2 $x_{;\nu}^{\mu} = \frac{\partial x^{\mu}}{\partial x^{\nu}}$ is established in tensor satisfying Binary Law.

Proof: If all coordinate systems satisfies Binary Law in $\frac{\partial g_{\nu\epsilon}}{\partial x^{\tau}} - \frac{\partial g_{\tau\nu}}{\partial x^{\epsilon}}$,

$$\frac{\partial g_{\nu\epsilon}}{\partial x^{\tau}} - \frac{\partial g_{\tau\nu}}{\partial x^{\epsilon}} = 0 \quad (8)$$

is established. I get

$$x_{;\nu}^{\mu} = \frac{\partial x^{\mu}}{\partial x^{\nu}} + x^{\tau} \frac{1}{2} g^{\epsilon\mu} \left(\frac{\partial g_{\tau\epsilon}}{\partial x^{\nu}} \right) = \frac{\partial x^{\mu}}{\partial x^{\nu}} + x^{\tau} \frac{1}{2} \frac{\partial g_{\tau}^{\mu}}{\partial x^{\nu}} \quad (9)$$

from (8),Definision9. I get

$$x_{;\nu}^{\mu} = \frac{\partial x^{\mu}}{\partial x^{\nu}} + x^{\tau} \frac{1}{2} \frac{\partial g_{\tau}^{\mu}}{\partial x^{\nu}} \quad (10)$$

as if Binary Law being satisfied for all index except the dummy index of (9). If (10) is a tensor equation, the dummy index of (10) can't make μ or ν . On the other hand, If all coordinate systems satisfies Binary Law, dummy index of (10) should be μ or ν in consideration of Definision5. I aim at the coexistence of these two demands. I rewrite (10) using Definision2 and get

$$x^{\mu;\mu} = \frac{\partial x^{\mu}}{\partial x_{\mu}} + x^{\tau} \frac{1}{2} \frac{\partial g_{\tau}^{\mu}}{\partial x_{\mu}} = \frac{\partial x^{\mu}}{\partial x_{\mu}} + x^{\nu} \frac{1}{2} \frac{\partial g_{\nu}^{\mu}}{\partial x_{\mu}}. \quad (11)$$

(11) satisfies two demands mentioned above together here. I get

$$x^{\mu;\mu} = \frac{\partial x^{\mu}}{\partial x_{\mu}} \quad (12)$$

in consideration of Definision6 for (11). Because the second term of the right side doesn't exist in (12),

$$x_{;\nu}^{\mu} = \frac{\partial x^{\mu}}{\partial x^{\nu}} \quad (13)$$

can rewrite (12) using Definision2. I rewrite (10) using Definision4 and get

$$-x_{;\mu}^{\mu} = -\frac{\partial x^{\mu}}{\partial x^{\mu}} - x^{\tau} \frac{1}{2} \frac{\partial g_{\tau}^{\mu}}{\partial x^{\mu}} = -\frac{\partial x^{\mu}}{\partial x^{\mu}} - x^{\nu} \frac{1}{2} \frac{\partial g_{\nu}^{\mu}}{\partial x^{\mu}}. \quad (14)$$

(14) satisfies two demands mentioned above together here. I get

$$-x_{;\mu}^{\mu} = -\frac{\partial x^{\mu}}{\partial x^{\mu}} \quad (15)$$

in consideration of Definision6 for (14). Because the second term of the right side doesn't exist in (15), (13) can rewrite (15) using Definision4.

If all coordinate systems satisfies Binary Law in $\frac{\partial g_{\tau\epsilon}}{\partial x^v} - \frac{\partial g_{\tau v}}{\partial x^\epsilon}$,

$$\frac{\partial g_{\tau\epsilon}}{\partial x^v} - \frac{\partial g_{\tau v}}{\partial x^\epsilon} = 0 \quad (16)$$

is established. I get

$$x_{;v}^\mu = \frac{\partial x^\mu}{\partial x^v} + x^\tau \frac{1}{2} g^{\epsilon\mu} \left(\frac{\partial g_{v\epsilon}}{\partial x^\tau} \right) = \frac{\partial x^\mu}{\partial x^v} + x^\tau \frac{1}{2} \left(\frac{\partial g_v^\mu}{\partial x^\tau} \right) \quad (17)$$

from (8),Definision9. I get

$$x_{;v}^\mu = \frac{\partial x^\mu}{\partial x^v} + x^\tau \frac{1}{2} \left(\frac{\partial g_v^\mu}{\partial x^\tau} \right) \quad (18)$$

as if Binary Law being satisfied for all index except the dummy index of (17). If (18) is a tensor equation, the dummy index of (18) can't make μ or v . On the other hand, If all coordinate systems satisfies Binary Law, dummy index of (18) should be μ or v in consideration of Definision5. I aim at the coexistence of these two demands. I rewrite (18) using Definision2 and get

$$x^{\mu;\mu} = \frac{\partial x^\mu}{\partial x_\mu} + x^\tau \frac{1}{2} \left(\frac{\partial g^{\mu\mu}}{\partial x^\tau} \right) = \frac{\partial x^\mu}{\partial x_\mu} + x^v \frac{1}{2} \left(\frac{\partial g^{\mu\mu}}{\partial x^v} \right). \quad (19)$$

(19) satisfies two demands mentioned above together here. I rewrite (18) using Definision4 and get

$$-x_{;\mu}^\mu = -\frac{\partial x^\mu}{\partial x^\mu} - x^\tau \frac{1}{2} \left(\frac{\partial g_\mu^\mu}{\partial x^\tau} \right) = -\frac{\partial x^\mu}{\partial x^\mu} - x^v \frac{1}{2} \left(\frac{\partial g_\mu^\mu}{\partial x^v} \right). \quad (20)$$

(20) satisfies two demands mentioned above together here. I get

$$-x_{;\mu}^\mu = -\frac{\partial x^\mu}{\partial x^\mu} \quad (21)$$

in consideration of Definision6 for (20). Because the second term of the right side doesn't exist in (21), (13) can rewrite (21) using Definision4. End Proof

Proposition3 $x_{\mu;v;\nu} = \frac{\partial^2 x_\mu}{\partial x^v \partial x^\nu}$ is established in tensor satisfying Binary Law.

Proof: If all coordinate systems satisfies Binary Law in

$$\left\{ \frac{\partial g_{\mu\epsilon}}{\partial x^v} - \frac{\partial g_{\mu v}}{\partial x^\epsilon}, \frac{\partial g_{\mu\epsilon}}{\partial x^\sigma} - \frac{\partial g_{\mu\sigma}}{\partial x^\epsilon}, \frac{\partial g_{v\epsilon}}{\partial x^\iota} - \frac{\partial g_{\iota v}}{\partial x^\epsilon}, \frac{\partial g_{\sigma\epsilon}}{\partial x^v} - \frac{\partial g_{v\sigma}}{\partial x^\epsilon}, \frac{\partial g_{\mu\epsilon}}{\partial x^\iota} - \frac{\partial g_{\iota\mu}}{\partial x^\epsilon} \right\},$$

$$\frac{\partial g_{\mu\epsilon}}{\partial x^v} - \frac{\partial g_{\mu v}}{\partial x^\epsilon} = 0, \frac{\partial g_{\mu\epsilon}}{\partial x^\sigma} - \frac{\partial g_{\mu\sigma}}{\partial x^\epsilon} = 0, \frac{\partial g_{v\epsilon}}{\partial x^\iota} - \frac{\partial g_{\iota v}}{\partial x^\epsilon} = 0,$$

$$\frac{\partial g_{\sigma\epsilon}}{\partial x^v} - \frac{\partial g_{v\sigma}}{\partial x^\epsilon} = 0, \frac{\partial g_{\mu\epsilon}}{\partial x^\iota} - \frac{\partial g_{\iota\mu}}{\partial x^\epsilon} = 0 \quad (22)$$

is established. I get

$$x_{\mu;v;\sigma} = \frac{\partial^2 x_\mu}{\partial x^v \partial x^\sigma} - \frac{\partial}{\partial x^\sigma} \left(x_\tau \frac{1}{2} g^{\epsilon\tau} \left(\frac{\partial g_{v\epsilon}}{\partial x^\mu} \right) \right) - \frac{\partial x_\iota}{\partial x^v} \frac{1}{2} g^{\epsilon\iota} \left(\frac{\partial g_{\sigma\epsilon}}{\partial x^\mu} \right)$$

$$+ x_\tau \frac{1}{2} g^{\epsilon\tau} \left(\frac{\partial g_{\iota\epsilon}}{\partial x^v} \right) \frac{1}{2} g^{\epsilon\iota} \left(\frac{\partial g_{\sigma\epsilon}}{\partial x^\mu} \right) - \frac{\partial x_\mu}{\partial x^\iota} \frac{1}{2} g^{\epsilon\iota} \left(\frac{\partial g_{v\epsilon}}{\partial x^\sigma} \right) + x_\tau \frac{1}{2} g^{\epsilon\tau} \left(\frac{\partial g_{\iota\epsilon}}{\partial x^\mu} \right) \frac{1}{2} g^{\epsilon\iota} \left(\frac{\partial g_{v\epsilon}}{\partial x^\sigma} \right)$$

$$= \frac{\partial^2 x_\mu}{\partial x^v \partial x^\sigma} - \frac{\partial}{\partial x^\sigma} \left(x_\tau \frac{1}{2} \frac{\partial g_v^\tau}{\partial x^\mu} \right) - \frac{\partial x_\iota}{\partial x^v} \frac{1}{2} \frac{\partial g_\sigma^\iota}{\partial x^\mu} + x_\tau \frac{1}{2} \frac{\partial g_\tau^\iota}{\partial x^v} \frac{1}{2} \frac{\partial g_\sigma^\iota}{\partial x^\mu} - \frac{\partial x_\mu}{\partial x^\iota} \frac{1}{2} \frac{\partial g_v^\tau}{\partial x^\sigma} + x_\tau \frac{1}{2} \frac{\partial g_\tau^\iota}{\partial x^\mu} \frac{1}{2} \frac{\partial g_v^\tau}{\partial x^\sigma} \quad (23)$$

from (22), Definision10. I get

$$x_{\mu;v;\nu} = \frac{\partial^2 x_\mu}{\partial x^\nu \partial x^\nu} - \frac{\partial}{\partial x^\nu} \left(x_\tau \frac{1}{2} \frac{\partial g_\tau^\nu}{\partial x^\mu} \right) - \frac{\partial x_\tau}{\partial x^\nu} \frac{1}{2} \frac{\partial g_\tau^\nu}{\partial x^\mu} + x_\tau \frac{1}{2} \frac{\partial g_\tau^\tau}{\partial x^\nu} \frac{1}{2} \frac{\partial g_\tau^\nu}{\partial x^\mu} - \frac{\partial x_\mu}{\partial x^\tau} \frac{1}{2} \frac{\partial g_\tau^\nu}{\partial x^\nu} + x_\tau \frac{1}{2} \frac{\partial g_\tau^\tau}{\partial x^\mu} \frac{1}{2} \frac{\partial g_\tau^\nu}{\partial x^\nu} \quad (24)$$

as if Binary Law being satisfied for all index except the dummy index of (23). If (24) is a tensor equation, the dummy index of (24) can't make μ or ν . On the other hand, If all coordinate systems satisfies Binary Law, dummy index of (24) should be μ or ν in consideration of Definision5. I aim at the coexistence of these two demands. I rewrite (24) using Definision2 and get

$$\begin{aligned} x_{\mu}^{;\mu;\mu} &= \frac{\partial^2 x_\mu}{\partial x_\mu \partial x_\mu} - \frac{\partial}{\partial x_\mu} \left(x_\tau \frac{1}{2} \frac{\partial g^{\mu\tau}}{\partial x^\mu} \right) - \frac{\partial x_\tau}{\partial x_\mu} \frac{1}{2} \frac{\partial g^{\mu\tau}}{\partial x^\mu} + x_\tau \frac{1}{2} \frac{\partial g_\tau^\tau}{\partial x_\mu} \frac{1}{2} \frac{\partial g^{\mu\tau}}{\partial x^\mu} - \frac{\partial x_\mu}{\partial x^\tau} \frac{1}{2} \frac{\partial g^{\mu\tau}}{\partial x^\mu} + x_\tau \frac{1}{2} \frac{\partial g_\tau^\tau}{\partial x^\mu} \frac{1}{2} \frac{\partial g^{\mu\tau}}{\partial x^\mu} \\ &= \frac{\partial^2 x_\mu}{\partial x_\mu \partial x_\mu} - \frac{\partial}{\partial x_\mu} \left(x_\nu \frac{1}{2} \frac{\partial g^{\mu\nu}}{\partial x^\mu} \right) - \frac{\partial x_\nu}{\partial x_\mu} \frac{1}{2} \frac{\partial g^{\mu\nu}}{\partial x^\mu} + x_\nu \frac{1}{2} \frac{\partial g_\nu^\nu}{\partial x_\mu} \frac{1}{2} \frac{\partial g^{\mu\nu}}{\partial x^\mu} - \frac{\partial x_\mu}{\partial x^\nu} \frac{1}{2} \frac{\partial g^{\mu\nu}}{\partial x^\mu} + x_\nu \frac{1}{2} \frac{\partial g_\nu^\nu}{\partial x^\mu} \frac{1}{2} \frac{\partial g^{\mu\nu}}{\partial x^\mu}. \end{aligned} \quad (25)$$

(25) satisfies two demands mentioned above together here. I rewrite (24) using Definision4 and get

$$\begin{aligned} x_{\mu;\mu;\mu} &= \frac{\partial^2 x_\mu}{\partial x^\mu \partial x^\mu} - \frac{\partial}{\partial x^\mu} \left(x_\tau \frac{1}{2} \frac{\partial g_\tau^\tau}{\partial x^\mu} \right) - \frac{\partial x_\tau}{\partial x^\mu} \frac{1}{2} \frac{\partial g_\tau^\tau}{\partial x^\mu} + x_\tau \frac{1}{2} \frac{\partial g_\tau^\tau}{\partial x^\mu} \frac{1}{2} \frac{\partial g_\tau^\tau}{\partial x^\mu} - \frac{\partial x_\mu}{\partial x^\tau} \frac{1}{2} \frac{\partial g_\tau^\tau}{\partial x^\mu} + x_\tau \frac{1}{2} \frac{\partial g_\tau^\tau}{\partial x^\mu} \frac{1}{2} \frac{\partial g_\tau^\tau}{\partial x^\mu} \\ &= \frac{\partial^2 x_\mu}{\partial x^\mu \partial x^\mu} - \frac{\partial}{\partial x^\mu} \left(x_\nu \frac{1}{2} \frac{\partial g_\nu^\nu}{\partial x^\mu} \right) - \frac{\partial x_\nu}{\partial x^\mu} \frac{1}{2} \frac{\partial g_\nu^\nu}{\partial x^\mu} + x_\nu \frac{1}{2} \frac{\partial g_\nu^\nu}{\partial x^\mu} \frac{1}{2} \frac{\partial g_\nu^\nu}{\partial x^\mu} - \frac{\partial x_\mu}{\partial x^\nu} \frac{1}{2} \frac{\partial g_\nu^\nu}{\partial x^\mu} + x_\nu \frac{1}{2} \frac{\partial g_\nu^\nu}{\partial x^\mu} \frac{1}{2} \frac{\partial g_\nu^\nu}{\partial x^\mu}. \end{aligned} \quad (26)$$

(26) satisfies two demands mentioned above together here. I get

$$x_{\mu;\mu;\mu} = \frac{\partial^2 x_\mu}{\partial x^\mu \partial x^\mu} \quad (27)$$

in consideration of Definision6 for (26). Because the second term of the right side doesn't exist in (27),

$$x_{\mu;v;\nu} = \frac{\partial^2 x_\mu}{\partial x^\nu \partial x^\nu} \quad (28)$$

can rewrite (27) using Definision4.

If all coordinate systems satisfies Binary Law in

$$\begin{aligned} &\left\{ \frac{\partial g_{\mu\epsilon}}{\partial x^\nu} - \frac{\partial g_{\mu\nu}}{\partial x^\epsilon}, \frac{\partial g_{\mu\epsilon}}{\partial x^\sigma} - \frac{\partial g_{\mu\sigma}}{\partial x^\epsilon}, \frac{\partial g_{\tau\epsilon}}{\partial x^\nu} - \frac{\partial g_{\tau\nu}}{\partial x^\epsilon}, \frac{\partial g_{\nu\epsilon}}{\partial x^\sigma} - \frac{\partial g_{\nu\sigma}}{\partial x^\epsilon}, \frac{\partial g_{\mu\epsilon}}{\partial x^\tau} - \frac{\partial g_{\mu\tau}}{\partial x^\epsilon} \right\}, \\ &\frac{\partial g_{\mu\epsilon}}{\partial x^\nu} - \frac{\partial g_{\mu\nu}}{\partial x^\epsilon} = 0, \frac{\partial g_{\mu\epsilon}}{\partial x^\sigma} - \frac{\partial g_{\mu\sigma}}{\partial x^\epsilon} = 0, \frac{\partial g_{\tau\epsilon}}{\partial x^\nu} - \frac{\partial g_{\tau\nu}}{\partial x^\epsilon} = 0, \\ &\frac{\partial g_{\nu\epsilon}}{\partial x^\sigma} - \frac{\partial g_{\nu\sigma}}{\partial x^\epsilon} = 0, \frac{\partial g_{\mu\epsilon}}{\partial x^\tau} - \frac{\partial g_{\mu\tau}}{\partial x^\epsilon} = 0 \end{aligned} \quad (29)$$

is established. I get

$$\begin{aligned} x_{\mu;v;\sigma} &= \frac{\partial^2 x_\mu}{\partial x^\nu \partial x^\sigma} - \frac{\partial}{\partial x^\sigma} \left(x_\tau \frac{1}{2} g^{\epsilon\tau} \left(\frac{\partial g_{\nu\epsilon}}{\partial x^\mu} \right) \right) - \frac{\partial x_\tau}{\partial x^\nu} \frac{1}{2} g^{\epsilon\tau} \left(\frac{\partial g_{\sigma\epsilon}}{\partial x^\mu} \right) \\ &+ x_\tau \frac{1}{2} g^{\epsilon\tau} \left(\frac{\partial g_{\nu\epsilon}}{\partial x^\tau} \right) \frac{1}{2} g^{\epsilon\tau} \left(\frac{\partial g_{\sigma\epsilon}}{\partial x^\mu} \right) - \frac{\partial x_\mu}{\partial x^\tau} \frac{1}{2} g^{\epsilon\tau} \left(\frac{\partial g_{\sigma\epsilon}}{\partial x^\nu} \right) + x_\tau \frac{1}{2} g^{\epsilon\tau} \left(\frac{\partial g_{\tau\epsilon}}{\partial x^\mu} \right) \frac{1}{2} g^{\epsilon\tau} \left(\frac{\partial g_{\sigma\epsilon}}{\partial x^\nu} \right) \\ &= \frac{\partial^2 x_\mu}{\partial x^\nu \partial x^\sigma} - \frac{\partial}{\partial x^\sigma} \left(x_\tau \frac{1}{2} \frac{\partial g_\tau^\nu}{\partial x^\mu} \right) - \frac{\partial x_\tau}{\partial x^\nu} \frac{1}{2} \frac{\partial g_\tau^\sigma}{\partial x^\mu} + x_\tau \frac{1}{2} \frac{\partial g_\tau^\tau}{\partial x^\tau} \frac{1}{2} \frac{\partial g_\tau^\sigma}{\partial x^\mu} - \frac{\partial x_\mu}{\partial x^\tau} \frac{1}{2} \frac{\partial g_\tau^\sigma}{\partial x^\nu} + x_\tau \frac{1}{2} \frac{\partial g_\tau^\tau}{\partial x^\mu} \frac{1}{2} \frac{\partial g_\tau^\sigma}{\partial x^\nu} \end{aligned} \quad (30)$$

from (29), Definision10. I get

$$x_{\mu;v;\nu} = \frac{\partial^2 x_\mu}{\partial x^\nu \partial x^\nu} - \frac{\partial}{\partial x^\nu} \left(x_\tau \frac{1}{2} \frac{\partial g_\tau^\nu}{\partial x^\mu} \right) - \frac{\partial x_\tau}{\partial x^\nu} \frac{1}{2} \frac{\partial g_\tau^\nu}{\partial x^\mu} + x_\tau \frac{1}{2} \frac{\partial g_\tau^\tau}{\partial x^\tau} \frac{1}{2} \frac{\partial g_\tau^\nu}{\partial x^\mu} - \frac{\partial x_\mu}{\partial x^\tau} \frac{1}{2} \frac{\partial g_\tau^\nu}{\partial x^\nu} + x_\tau \frac{1}{2} \frac{\partial g_\tau^\tau}{\partial x^\mu} \frac{1}{2} \frac{\partial g_\tau^\nu}{\partial x^\nu} \quad (31)$$

as if Binary Law being satisfied for all index except the dummy index of (30). If (31) is a tensor equation, the dummy index of (31) can't make μ or ν . On the other hand, If all coordinate

systems satisfies Binary Law, dummy index of (31) should be μ or ν in consideration of Definision5. I aim at the coexistence of these two demands. I rewrite (31) using Definision2 and get

$$\begin{aligned} x_{\mu}^{\mu;\mu} &= \frac{\partial^2 x_{\mu}}{\partial x_{\mu} \partial x_{\mu}} - \frac{\partial}{\partial x_{\mu}} \left(x_{\tau} \frac{1}{2} \frac{\partial g^{\mu\tau}}{\partial x^{\mu}} \right) - \frac{\partial x_{\tau}}{\partial x_{\mu}} \frac{1}{2} \frac{\partial g^{\mu\tau}}{\partial x^{\mu}} + x_{\tau} \frac{1}{2} \frac{\partial g^{\mu\tau}}{\partial x^{\tau}} \frac{1}{2} \frac{\partial g^{\mu\mu}}{\partial x^{\mu}} - \frac{\partial x_{\mu}}{\partial x^{\tau}} \frac{1}{2} \frac{\partial g^{\mu\mu}}{\partial x_{\mu}} + x_{\tau} \frac{1}{2} \frac{\partial g_{\tau}^{\tau}}{\partial x^{\mu}} \frac{1}{2} \frac{\partial g^{\mu\mu}}{\partial x_{\mu}} \\ &= \frac{\partial^2 x_{\mu}}{\partial x_{\mu} \partial x_{\mu}} - \frac{\partial}{\partial x_{\mu}} \left(x_{\nu} \frac{1}{2} \frac{\partial g^{\mu\nu}}{\partial x^{\mu}} \right) - \frac{\partial x_{\nu}}{\partial x_{\mu}} \frac{1}{2} \frac{\partial g^{\mu\nu}}{\partial x^{\mu}} + x_{\nu} \frac{1}{2} \frac{\partial g^{\mu\nu}}{\partial x^{\nu}} \frac{1}{2} \frac{\partial g^{\mu\nu}}{\partial x^{\mu}} - \frac{\partial x_{\mu}}{\partial x^{\nu}} \frac{1}{2} \frac{\partial g^{\mu\nu}}{\partial x_{\mu}} + x_{\nu} \frac{1}{2} \frac{\partial g_{\nu}^{\nu}}{\partial x^{\mu}} \frac{1}{2} \frac{\partial g^{\mu\nu}}{\partial x_{\mu}}. \end{aligned} \quad (32)$$

(32) satisfies two demands mentioned above together here. I rewrite (31) using Definision4 and get

$$\begin{aligned} x_{\mu;\mu;\mu} &= \frac{\partial^2 x_{\mu}}{\partial x^{\mu} \partial x^{\mu}} - \frac{\partial}{\partial x^{\mu}} \left(x_{\tau} \frac{1}{2} \frac{\partial g_{\tau}^{\tau}}{\partial x^{\mu}} \right) - \frac{\partial x_{\tau}}{\partial x^{\mu}} \frac{1}{2} \frac{\partial g_{\tau}^{\tau}}{\partial x^{\mu}} + x_{\tau} \frac{1}{2} \frac{\partial g_{\tau}^{\tau}}{\partial x^{\tau}} \frac{1}{2} \frac{\partial g_{\tau}^{\tau}}{\partial x^{\mu}} - \frac{\partial x_{\mu}}{\partial x^{\tau}} \frac{1}{2} \frac{\partial g_{\tau}^{\tau}}{\partial x^{\mu}} + x_{\tau} \frac{1}{2} \frac{\partial g_{\tau}^{\tau}}{\partial x^{\mu}} \frac{1}{2} \frac{\partial g_{\tau}^{\tau}}{\partial x^{\mu}} \\ &= \frac{\partial^2 x_{\mu}}{\partial x^{\mu} \partial x^{\mu}} - \frac{\partial}{\partial x^{\mu}} \left(x_{\nu} \frac{1}{2} \frac{\partial g_{\nu}^{\nu}}{\partial x^{\mu}} \right) - \frac{\partial x_{\nu}}{\partial x^{\mu}} \frac{1}{2} \frac{\partial g_{\nu}^{\nu}}{\partial x^{\mu}} + x_{\nu} \frac{1}{2} \frac{\partial g_{\nu}^{\nu}}{\partial x^{\nu}} \frac{1}{2} \frac{\partial g_{\nu}^{\nu}}{\partial x^{\mu}} - \frac{\partial x_{\mu}}{\partial x^{\nu}} \frac{1}{2} \frac{\partial g_{\nu}^{\nu}}{\partial x^{\mu}} + x_{\nu} \frac{1}{2} \frac{\partial g_{\nu}^{\nu}}{\partial x^{\mu}} \frac{1}{2} \frac{\partial g_{\nu}^{\nu}}{\partial x^{\mu}}. \end{aligned} \quad (33)$$

(33) satisfies two demands mentioned above together here. I get

$$x_{\mu;\mu;\mu} = \frac{\partial^2 x_{\mu}}{\partial x^{\mu} \partial x^{\mu}} \quad (34)$$

in consideration of Definision6 for (33). Because the second term of the right side doesn't exist in (34), (28) can rewrite (34) using Definision4. End Proof

Proposition4 $x_{\nu;\nu}^{\mu} = \frac{\partial^2 x^{\mu}}{\partial x^{\nu} \partial x^{\nu}}$ is established in tensor satisfying Binary Law.

Proof: If all coordinate systems satisfies Binary Law in

$$\left\{ \frac{\partial g_{\nu\epsilon}}{\partial x^{\tau}} - \frac{\partial g_{\tau\nu}}{\partial x^{\epsilon}}, \frac{\partial g_{\sigma\epsilon}}{\partial x^{\tau}} - \frac{\partial g_{\tau\sigma}}{\partial x^{\epsilon}}, \frac{\partial g_{\nu\sigma}}{\partial x^{\epsilon}} - \frac{\partial g_{\sigma\nu}}{\partial x^{\epsilon}}, \frac{\partial g_{\tau\epsilon}}{\partial x^{\sigma}} - \frac{\partial g_{\sigma\tau}}{\partial x^{\epsilon}} \right\},$$

$$\frac{\partial g_{\nu\epsilon}}{\partial x^{\tau}} - \frac{\partial g_{\tau\nu}}{\partial x^{\epsilon}} = 0, \frac{\partial g_{\sigma\epsilon}}{\partial x^{\tau}} - \frac{\partial g_{\tau\sigma}}{\partial x^{\epsilon}} = 0, \frac{\partial g_{\nu\sigma}}{\partial x^{\epsilon}} - \frac{\partial g_{\sigma\nu}}{\partial x^{\epsilon}} = 0, \frac{\partial g_{\tau\epsilon}}{\partial x^{\sigma}} - \frac{\partial g_{\sigma\tau}}{\partial x^{\epsilon}} = 0 \quad (35)$$

is established. I get

$$\begin{aligned} x_{\nu;\nu}^{\mu} &= \frac{\partial^2 x^{\mu}}{\partial x^{\nu} \partial x^{\nu}} + \frac{\partial}{\partial x^{\sigma}} \left(x^{\tau} \frac{1}{2} g^{\epsilon\mu} \left(\frac{\partial g_{\tau\epsilon}}{\partial x^{\nu}} \right) \right) + \frac{\partial x^{\tau}}{\partial x^{\nu}} \frac{1}{2} g^{\epsilon\mu} \left(\frac{\partial g_{\tau\epsilon}}{\partial x^{\sigma}} \right) \\ &+ x^{\tau} \frac{1}{2} g^{\epsilon\mu} \left(\frac{\partial g_{\tau\epsilon}}{\partial x^{\nu}} \right) \frac{1}{2} g^{\epsilon\mu} \left(\frac{\partial g_{\tau\epsilon}}{\partial x^{\sigma}} \right) - \frac{\partial x^{\mu}}{\partial x^{\tau}} \frac{1}{2} g^{\epsilon\mu} \left(\frac{\partial g_{\nu\epsilon}}{\partial x^{\sigma}} \right) - x^{\tau} \frac{1}{2} g^{\epsilon\mu} \left(\frac{\partial g_{\tau\epsilon}}{\partial x^{\tau}} \right) \frac{1}{2} g^{\epsilon\mu} \left(\frac{\partial g_{\nu\epsilon}}{\partial x^{\sigma}} \right) \\ &= \frac{\partial^2 x^{\mu}}{\partial x^{\nu} \partial x^{\nu}} + \frac{\partial}{\partial x^{\sigma}} \left(x^{\tau} \frac{1}{2} \frac{\partial g_{\tau}^{\mu}}{\partial x^{\nu}} \right) + \frac{\partial x^{\tau}}{\partial x^{\nu}} \frac{1}{2} \frac{\partial g_{\tau}^{\mu}}{\partial x^{\sigma}} + x^{\tau} \frac{1}{2} \frac{\partial g_{\tau}^{\mu}}{\partial x^{\nu}} \frac{1}{2} \frac{\partial g_{\tau}^{\mu}}{\partial x^{\sigma}} - \frac{\partial x^{\mu}}{\partial x^{\tau}} \frac{1}{2} \frac{\partial g_{\nu}^{\mu}}{\partial x^{\sigma}} - x^{\tau} \frac{1}{2} \frac{\partial g_{\tau}^{\mu}}{\partial x^{\tau}} \frac{1}{2} \frac{\partial g_{\nu}^{\mu}}{\partial x^{\sigma}} \end{aligned} \quad (36)$$

from (35), Definision11. I get

$$x_{\nu;\nu}^{\mu} = \frac{\partial^2 x^{\mu}}{\partial x^{\nu} \partial x^{\nu}} + \frac{\partial}{\partial x^{\nu}} \left(x^{\tau} \frac{1}{2} \frac{\partial g_{\tau}^{\mu}}{\partial x^{\nu}} \right) + \frac{\partial x^{\tau}}{\partial x^{\nu}} \frac{1}{2} \frac{\partial g_{\tau}^{\mu}}{\partial x^{\nu}} + x^{\tau} \frac{1}{2} \frac{\partial g_{\tau}^{\mu}}{\partial x^{\nu}} \frac{1}{2} \frac{\partial g_{\tau}^{\mu}}{\partial x^{\nu}} - \frac{\partial x^{\mu}}{\partial x^{\tau}} \frac{1}{2} \frac{\partial g_{\nu}^{\mu}}{\partial x^{\nu}} - x^{\tau} \frac{1}{2} \frac{\partial g_{\tau}^{\mu}}{\partial x^{\tau}} \frac{1}{2} \frac{\partial g_{\nu}^{\mu}}{\partial x^{\nu}} \quad (37)$$

as if Binary Law being satisfied for all index except the dummy index of (36). If (37) is a tensor equation, the dummy index of (37) can't make μ or ν . On the other hand, If all coordinate systems satisfies Binary Law, dummy index of (37) should be μ or ν in consideration of Definision5. I aim at the coexistence of these two demands. I rewrite (37) using Definision2 and get

$$\begin{aligned} x_{\mu;\mu;\mu} &= \frac{\partial^2 x^{\mu}}{\partial x_{\mu} \partial x_{\mu}} + \frac{\partial}{\partial x_{\mu}} \left(x^{\tau} \frac{1}{2} \frac{\partial g_{\tau}^{\mu}}{\partial x_{\mu}} \right) + \frac{\partial x^{\tau}}{\partial x_{\mu}} \frac{1}{2} \frac{\partial g_{\tau}^{\mu}}{\partial x_{\mu}} + x^{\tau} \frac{1}{2} \frac{\partial g_{\tau}^{\mu}}{\partial x_{\mu}} \frac{1}{2} \frac{\partial g_{\tau}^{\mu}}{\partial x_{\mu}} - \frac{\partial x^{\mu}}{\partial x^{\tau}} \frac{1}{2} \frac{\partial g^{\mu\mu}}{\partial x_{\mu}} - x^{\tau} \frac{1}{2} \frac{\partial g_{\tau}^{\mu}}{\partial x^{\tau}} \frac{1}{2} \frac{\partial g^{\mu\mu}}{\partial x_{\mu}} \\ &= \frac{\partial^2 x^{\mu}}{\partial x_{\mu} \partial x_{\mu}} + \frac{\partial}{\partial x_{\mu}} \left(x^{\nu} \frac{1}{2} \frac{\partial g_{\nu}^{\mu}}{\partial x_{\mu}} \right) + \frac{\partial x^{\nu}}{\partial x_{\mu}} \frac{1}{2} \frac{\partial g_{\nu}^{\mu}}{\partial x_{\mu}} + x^{\nu} \frac{1}{2} \frac{\partial g_{\nu}^{\mu}}{\partial x_{\mu}} \frac{1}{2} \frac{\partial g_{\nu}^{\mu}}{\partial x_{\mu}} - \frac{\partial x^{\mu}}{\partial x^{\nu}} \frac{1}{2} \frac{\partial g^{\mu\nu}}{\partial x_{\mu}} - x^{\nu} \frac{1}{2} \frac{\partial g_{\nu}^{\mu}}{\partial x^{\nu}} \frac{1}{2} \frac{\partial g^{\mu\nu}}{\partial x_{\mu}}. \end{aligned} \quad (38)$$

(38) satisfies two demands mentioned above together here. I get

$$x^{\mu;\mu;\mu} = \frac{\partial^2 x^\mu}{\partial x_\mu \partial x_\mu} - \frac{\partial x^\mu}{\partial x^\nu} \frac{1}{2} \frac{\partial g^{\mu\nu}}{\partial x_\mu} \quad (39)$$

in consideration of Definision6 for (38). I rewrite (37) using Definision4 and get

$$\begin{aligned} x^{\mu}_{;\mu;\mu} &= \frac{\partial^2 x^\mu}{\partial x^\mu \partial x^\mu} + \frac{\partial}{\partial x^\mu} \left(x^\tau \frac{1}{2} \frac{\partial g^\mu_\tau}{\partial x^\mu} \right) + \frac{\partial x^\mu}{\partial x^\mu} \frac{1}{2} \frac{\partial g^\mu_\mu}{\partial x^\mu} + x^\tau \frac{1}{2} \frac{\partial g^\mu_\tau}{\partial x^\mu} \frac{1}{2} \frac{\partial g^\mu_\mu}{\partial x^\mu} - \frac{\partial x^\mu}{\partial x^\mu} \frac{1}{2} \frac{\partial g^\mu_\mu}{\partial x^\mu} - x^\tau \frac{1}{2} \frac{\partial g^\mu_\tau}{\partial x^\mu} \frac{1}{2} \frac{\partial g^\mu_\mu}{\partial x^\mu} \\ &= \frac{\partial^2 x^\mu}{\partial x^\mu \partial x^\mu} + \frac{\partial}{\partial x^\mu} \left(x^\nu \frac{1}{2} \frac{\partial g^\mu_\nu}{\partial x^\mu} \right) + \frac{\partial x^\mu}{\partial x^\mu} \frac{1}{2} \frac{\partial g^\mu_\mu}{\partial x^\mu} + x^\nu \frac{1}{2} \frac{\partial g^\mu_\nu}{\partial x^\mu} \frac{1}{2} \frac{\partial g^\mu_\mu}{\partial x^\mu} - \frac{\partial x^\mu}{\partial x^\nu} \frac{1}{2} \frac{\partial g^\mu_\mu}{\partial x^\mu} - x^\nu \frac{1}{2} \frac{\partial g^\mu_\nu}{\partial x^\mu} \frac{1}{2} \frac{\partial g^\mu_\mu}{\partial x^\mu}. \end{aligned} \quad (40)$$

(40) satisfies two demands mentioned above together here. I get

$$x^{\mu}_{;\mu;\mu} = \frac{\partial^2 x^\mu}{\partial x^\mu \partial x^\mu} \quad (41)$$

in consideration of Definision6 for (40). Because the second term of the right side doesn't exist in (41),

$$x^{\mu}_{;\nu;\nu} = \frac{\partial^2 x^\mu}{\partial x^\nu \partial x^\nu} \quad (42)$$

can rewrite (41) using Definision4.

If all coordinate systems satisfies Binary Law in

$$\left\{ \frac{\partial g_{\tau\epsilon}}{\partial x^\nu} - \frac{\partial g_{\tau\nu}}{\partial x^\epsilon}, \frac{\partial g_{\iota\epsilon}}{\partial x^\sigma} - \frac{\partial g_{\iota\sigma}}{\partial x^\epsilon}, \frac{\partial g_{\nu\epsilon}}{\partial x^\sigma} - \frac{\partial g_{\nu\sigma}}{\partial x^\epsilon}, \frac{\partial g_{\tau\epsilon}}{\partial x^\iota} - \frac{\partial g_{\tau\iota}}{\partial x^\epsilon} \right\},$$

$$\frac{\partial g_{\tau\epsilon}}{\partial x^\nu} - \frac{\partial g_{\tau\nu}}{\partial x^\epsilon} = 0, \frac{\partial g_{\iota\epsilon}}{\partial x^\sigma} - \frac{\partial g_{\iota\sigma}}{\partial x^\epsilon} = 0, \frac{\partial g_{\nu\epsilon}}{\partial x^\sigma} - \frac{\partial g_{\nu\sigma}}{\partial x^\epsilon} = 0, \frac{\partial g_{\tau\epsilon}}{\partial x^\iota} - \frac{\partial g_{\tau\iota}}{\partial x^\epsilon} = 0 \quad (43)$$

is established. I get

$$\begin{aligned} x^{\mu}_{;\nu;\sigma} &= \frac{\partial^2 x^\mu}{\partial x^\nu \partial x^\sigma} + \frac{\partial}{\partial x^\sigma} \left(x^\tau \frac{1}{2} g^{\epsilon\mu} \left(\frac{\partial g_{\nu\epsilon}}{\partial x^\tau} \right) \right) + \frac{\partial x^\mu}{\partial x^\nu} \frac{1}{2} g^{\epsilon\mu} \left(\frac{\partial g_{\sigma\epsilon}}{\partial x^\iota} \right) \\ &+ x^\tau \frac{1}{2} g^{\epsilon\iota} \left(\frac{\partial g_{\nu\epsilon}}{\partial x^\tau} \right) \frac{1}{2} g^{\epsilon\mu} \left(\frac{\partial g_{\sigma\epsilon}}{\partial x^\iota} \right) - \frac{\partial x^\mu}{\partial x^\iota} \frac{1}{2} g^{\epsilon\iota} \left(\frac{\partial g_{\sigma\epsilon}}{\partial x^\nu} \right) - x^\tau \frac{1}{2} g^{\epsilon\mu} \left(\frac{\partial g_{\iota\epsilon}}{\partial x^\tau} \right) \frac{1}{2} g^{\epsilon\iota} \left(\frac{\partial g_{\sigma\epsilon}}{\partial x^\nu} \right) \\ &= \frac{\partial^2 x^\mu}{\partial x^\nu \partial x^\sigma} + \frac{\partial}{\partial x^\sigma} \left(x^\tau \frac{1}{2} \frac{\partial g^\mu_\nu}{\partial x^\tau} \right) + \frac{\partial x^\mu}{\partial x^\nu} \frac{1}{2} \frac{\partial g^\mu_\sigma}{\partial x^\iota} + x^\tau \frac{1}{2} \frac{\partial g^\mu_\nu}{\partial x^\tau} \frac{1}{2} \frac{\partial g^\mu_\sigma}{\partial x^\iota} - \frac{\partial x^\mu}{\partial x^\iota} \frac{1}{2} \frac{\partial g^\mu_\sigma}{\partial x^\nu} - x^\tau \frac{1}{2} \frac{\partial g^\mu_\tau}{\partial x^\iota} \frac{1}{2} \frac{\partial g^\mu_\sigma}{\partial x^\nu} \end{aligned} \quad (44)$$

from (43),Definision11. I get

$$x^{\mu}_{;\nu;\nu} = \frac{\partial^2 x^\mu}{\partial x^\nu \partial x^\nu} + \frac{\partial}{\partial x^\nu} \left(x^\tau \frac{1}{2} \frac{\partial g^\mu_\nu}{\partial x^\tau} \right) + \frac{\partial x^\mu}{\partial x^\nu} \frac{1}{2} \frac{\partial g^\mu_\nu}{\partial x^\iota} + x^\tau \frac{1}{2} \frac{\partial g^\mu_\nu}{\partial x^\tau} \frac{1}{2} \frac{\partial g^\mu_\nu}{\partial x^\iota} - \frac{\partial x^\mu}{\partial x^\iota} \frac{1}{2} \frac{\partial g^\mu_\nu}{\partial x^\nu} - x^\tau \frac{1}{2} \frac{\partial g^\mu_\tau}{\partial x^\iota} \frac{1}{2} \frac{\partial g^\mu_\nu}{\partial x^\nu} \quad (45)$$

as if Binary Law being satisfied for all index except the dummy index of (44). If (45) is a tensor equation, the dummy index of (45) can't make μ or ν . On the other hand, If all coordinate systems satisfies Binary Law, dummy index of (45) should be μ or ν in consideration of Definision5. I aim at the coexistence of these two demands. I rewrite (45) using Definision2 and get

$$\begin{aligned} x^{\mu;\mu;\mu} &= \frac{\partial^2 x^\mu}{\partial x_\mu \partial x_\mu} + \frac{\partial}{\partial x_\mu} \left(x^\tau \frac{1}{2} \frac{\partial g^{\mu\mu}}{\partial x^\tau} \right) + \frac{\partial x^\mu}{\partial x_\mu} \frac{1}{2} \frac{\partial g^{\mu\mu}}{\partial x^\iota} + x^\tau \frac{1}{2} \frac{\partial g^{\mu\mu}}{\partial x^\tau} \frac{1}{2} \frac{\partial g^{\mu\mu}}{\partial x^\iota} - \frac{\partial x^\mu}{\partial x^\iota} \frac{1}{2} \frac{\partial g^{\mu\mu}}{\partial x_\mu} - x^\tau \frac{1}{2} \frac{\partial g^\mu_\tau}{\partial x^\iota} \frac{1}{2} \frac{\partial g^{\mu\mu}}{\partial x_\mu} \\ &= \frac{\partial^2 x^\mu}{\partial x_\mu \partial x_\mu} + \frac{\partial}{\partial x_\mu} \left(x^\nu \frac{1}{2} \frac{\partial g^{\mu\mu}}{\partial x^\nu} \right) + \frac{\partial x^\mu}{\partial x_\mu} \frac{1}{2} \frac{\partial g^{\mu\mu}}{\partial x^\nu} + x^\nu \frac{1}{2} \frac{\partial g^{\mu\mu}}{\partial x^\nu} \frac{1}{2} \frac{\partial g^{\mu\mu}}{\partial x^\nu} - \frac{\partial x^\mu}{\partial x^\nu} \frac{1}{2} \frac{\partial g^{\mu\mu}}{\partial x_\mu} - x^\nu \frac{1}{2} \frac{\partial g^\mu_\nu}{\partial x^\nu} \frac{1}{2} \frac{\partial g^{\mu\mu}}{\partial x_\mu}. \end{aligned} \quad (46)$$

(46) satisfies two demands mentioned above together here. I rewrite (45) using Definision4 and get

$$x^\mu_{;\mu;\mu} = \frac{\partial^2 x^\mu}{\partial x^\mu \partial x^\mu} \quad (48)$$

Proposition5 $x_{i;v;v;v}^{\mu} = \frac{\partial^3 x^{\mu}}{\partial x^v \partial x^v \partial x^v}$ is established in tensor satisfying Binary Law.

$$\left\{ \begin{aligned} & \left\{ \frac{\partial g_{v\epsilon}}{\partial x^\tau} - \frac{\partial g_{tv}}{\partial x^\epsilon}, \frac{\partial g_{\sigma\epsilon}}{\partial x^l} - \frac{\partial g_{l\sigma}}{\partial x^\epsilon}, \frac{\partial g_{\sigma\epsilon}}{\partial x^v} - \frac{\partial g_{v\sigma}}{\partial x^\epsilon}, \frac{\partial g_{l\epsilon}}{\partial x^\tau} - \frac{\partial g_{\tau l}}{\partial x^\epsilon}, \frac{\partial g_{l\epsilon}}{\partial x^k} - \frac{\partial g_{k l}}{\partial x^\epsilon}, \frac{\partial g_{l\epsilon}}{\partial x^v} - \frac{\partial g_{v l}}{\partial x^\epsilon}, \frac{\partial g_{k\epsilon}}{\partial x^\tau} - \frac{\partial g_{\tau k}}{\partial x^\epsilon} \right\}, \\ & \left\{ \frac{\partial g_{\sigma\epsilon}}{\partial x^k} - \frac{\partial g_{k\sigma}}{\partial x^\epsilon}, \frac{\partial g_{l\epsilon}}{\partial x^\sigma} - \frac{\partial g_{\sigma l}}{\partial x^\epsilon}, \frac{\partial g_{k\epsilon}}{\partial x^l} - \frac{\partial g_{l k}}{\partial x^\epsilon}, \frac{\partial g_{k\epsilon}}{\partial x^v} - \frac{\partial g_{v k}}{\partial x^\epsilon} \right\} \end{aligned} \right\}$$

$$\begin{aligned} \frac{\partial g_{v\epsilon}}{\partial x^\tau} - \frac{\partial g_{tv}}{\partial x^\epsilon} &= 0, \frac{\partial g_{\sigma\epsilon}}{\partial x^l} - \frac{\partial g_{l\sigma}}{\partial x^\epsilon} = 0, \frac{\partial g_{\sigma\epsilon}}{\partial x^v} - \frac{\partial g_{v\sigma}}{\partial x^\epsilon} = 0, \frac{\partial g_{l\epsilon}}{\partial x^\tau} - \frac{\partial g_{\tau l}}{\partial x^\epsilon} = 0, \frac{\partial g_{l\epsilon}}{\partial x^k} - \frac{\partial g_{k l}}{\partial x^\epsilon} = 0, \frac{\partial g_{l\epsilon}}{\partial x^v} - \frac{\partial g_{v l}}{\partial x^\epsilon} = 0, \\ \frac{\partial g_{k\epsilon}}{\partial x^\tau} - \frac{\partial g_{\tau k}}{\partial x^\epsilon} &= 0, \frac{\partial g_{\sigma\epsilon}}{\partial x^k} - \frac{\partial g_{k\sigma}}{\partial x^\epsilon} = 0, \frac{\partial g_{l\epsilon}}{\partial x^\sigma} - \frac{\partial g_{\sigma l}}{\partial x^\epsilon} = 0, \frac{\partial g_{k\epsilon}}{\partial x^l} - \frac{\partial g_{l k}}{\partial x^\epsilon} = 0, \frac{\partial g_{k\epsilon}}{\partial x^v} - \frac{\partial g_{v k}}{\partial x^\epsilon} = 0 \end{aligned} \quad (49)$$
[illegible]

$$\begin{aligned}
& +x^\tau \frac{1}{2} \frac{\partial g_\tau^\mu}{\partial x^\tau} \frac{1}{2} \frac{\partial g_k^\sigma}{\partial x^\sigma} \frac{1}{2} \frac{\partial g_\lambda^\kappa}{\partial x^\lambda} - \frac{\partial^2 x^\mu}{\partial x^\nu \partial x^\kappa} \frac{1}{2} \frac{\partial g_\sigma^\kappa}{\partial x^\lambda} - \frac{\partial}{\partial x^\kappa} \left(x^\tau \frac{1}{2} \frac{\partial g_\tau^\mu}{\partial x^\nu} \right) \frac{1}{2} \frac{\partial g_\sigma^\kappa}{\partial x^\lambda} - \frac{\partial x^\iota}{\partial x^\nu} \frac{1}{2} \frac{\partial g_\iota^\mu}{\partial x^\kappa} \frac{1}{2} \frac{\partial g_\sigma^\kappa}{\partial x^\lambda} \\
& - x^\tau \frac{1}{2} \frac{\partial g_\tau^\iota}{\partial x^\nu} \frac{1}{2} \frac{\partial g_\iota^\mu}{\partial x^\kappa} \frac{1}{2} \frac{\partial g_\sigma^\kappa}{\partial x^\lambda} + \frac{\partial x^\mu}{\partial x^\iota} \frac{1}{2} \frac{\partial g_\nu^\iota}{\partial x^\kappa} \frac{1}{2} \frac{\partial g_\sigma^\kappa}{\partial x^\lambda} + x^\tau \frac{1}{2} \frac{\partial g_\tau^\mu}{\partial x^\iota} \frac{1}{2} \frac{\partial g_\nu^\iota}{\partial x^\kappa} \frac{1}{2} \frac{\partial g_\sigma^\kappa}{\partial x^\lambda} \quad (50)
\end{aligned}$$

from (49), Definition 12. I get

$$\begin{aligned}
x_{;v,v,v}^{\mu} = & \frac{\partial^3 x^{\mu}}{\partial x^v \partial x^v \partial x^v} + \frac{\partial^2}{\partial x^v \partial x^v} \left(x^{\tau} \frac{1}{2} \frac{\partial g_{\tau}^{\mu}}{\partial x^v} \right) + \frac{\partial}{\partial x^v} \left(\frac{\partial x^{\tau}}{\partial x^v} \frac{1}{2} \frac{\partial g_{\tau}^{\mu}}{\partial x^v} \right) + \frac{\partial}{\partial x^v} \left(x^{\tau} \frac{1}{2} \frac{\partial g_{\tau}^{\mu}}{\partial x^v} \frac{1}{2} \frac{\partial g_{\tau}^{\mu}}{\partial x^v} \right) \\
& - \frac{\partial}{\partial x^v} \left(\frac{\partial x^{\mu}}{\partial x^{\tau}} \frac{1}{2} \frac{\partial g_{\tau}^{\nu}}{\partial x^v} \right) - \frac{\partial}{\partial x^v} \left(x^{\tau} \frac{1}{2} \frac{\partial g_{\tau}^{\mu}}{\partial x^{\tau}} \frac{1}{2} \frac{\partial g_{\tau}^{\nu}}{\partial x^v} \right) + \frac{\partial^2 x^{\kappa}}{\partial x^v \partial x^v} \frac{1}{2} \frac{\partial g_{\tau}^{\mu}}{\partial x^v} + \frac{\partial}{\partial x^v} \left(x^{\tau} \frac{1}{2} \frac{\partial g_{\tau}^{\kappa}}{\partial x^v} \right) \frac{1}{2} \frac{\partial g_{\tau}^{\mu}}{\partial x^v} \\
& + \frac{\partial x^{\tau}}{\partial x^v} \frac{1}{2} \frac{\partial g_{\tau}^{\kappa}}{\partial x^v} \frac{1}{2} \frac{\partial g_{\tau}^{\mu}}{\partial x^v} + x^{\tau} \frac{1}{2} \frac{\partial g_{\tau}^{\kappa}}{\partial x^v} \frac{1}{2} \frac{\partial g_{\tau}^{\mu}}{\partial x^v} \frac{1}{2} \frac{\partial g_{\tau}^{\mu}}{\partial x^v} - \frac{\partial x^{\kappa}}{\partial x^{\tau}} \frac{1}{2} \frac{\partial g_{\tau}^{\nu}}{\partial x^v} \frac{1}{2} \frac{\partial g_{\tau}^{\mu}}{\partial x^v} - x^{\tau} \frac{1}{2} \frac{\partial g_{\tau}^{\kappa}}{\partial x^{\tau}} \frac{1}{2} \frac{\partial g_{\tau}^{\nu}}{\partial x^v} \frac{1}{2} \frac{\partial g_{\tau}^{\mu}}{\partial x^v} \\
& - \frac{\partial^2 x^{\mu}}{\partial x^{\kappa} \partial x^v} \frac{1}{2} \frac{\partial g_{\tau}^{\nu}}{\partial x^v} - \frac{\partial}{\partial x^v} \left(x^{\tau} \frac{1}{2} \frac{\partial g_{\tau}^{\mu}}{\partial x^{\kappa}} \right) \frac{1}{2} \frac{\partial g_{\tau}^{\nu}}{\partial x^v} - \frac{\partial x^{\tau}}{\partial x^{\kappa}} \frac{1}{2} \frac{\partial g_{\tau}^{\mu}}{\partial x^{\tau}} \frac{1}{2} \frac{\partial g_{\tau}^{\nu}}{\partial x^v} - x^{\tau} \frac{1}{2} \frac{\partial g_{\tau}^{\kappa}}{\partial x^{\tau}} \frac{1}{2} \frac{\partial g_{\tau}^{\mu}}{\partial x^{\tau}} \frac{1}{2} \frac{\partial g_{\tau}^{\nu}}{\partial x^v} \\
& + \frac{\partial x^{\mu}}{\partial x^{\tau}} \frac{1}{2} \frac{\partial g_{\tau}^{\kappa}}{\partial x^v} \frac{1}{2} \frac{\partial g_{\tau}^{\nu}}{\partial x^v} + x^{\tau} \frac{1}{2} \frac{\partial g_{\tau}^{\mu}}{\partial x^{\tau}} \frac{1}{2} \frac{\partial g_{\tau}^{\kappa}}{\partial x^v} \frac{1}{2} \frac{\partial g_{\tau}^{\nu}}{\partial x^v} - \frac{\partial^2 x^{\mu}}{\partial x^v \partial x^{\kappa}} \frac{1}{2} \frac{\partial g_{\tau}^{\nu}}{\partial x^v} - \frac{\partial}{\partial x^{\kappa}} \left(x^{\tau} \frac{1}{2} \frac{\partial g_{\tau}^{\mu}}{\partial x^v} \right) \frac{1}{2} \frac{\partial g_{\tau}^{\nu}}{\partial x^v} \\
& - \frac{\partial x^{\tau}}{\partial x^v} \frac{1}{2} \frac{\partial g_{\tau}^{\mu}}{\partial x^{\tau}} \frac{1}{2} \frac{\partial g_{\tau}^{\nu}}{\partial x^v} - x^{\tau} \frac{1}{2} \frac{\partial g_{\tau}^{\kappa}}{\partial x^{\tau}} \frac{1}{2} \frac{\partial g_{\tau}^{\mu}}{\partial x^{\tau}} \frac{1}{2} \frac{\partial g_{\tau}^{\nu}}{\partial x^v} + \frac{\partial x^{\mu}}{\partial x^{\tau}} \frac{1}{2} \frac{\partial g_{\tau}^{\nu}}{\partial x^{\tau}} \frac{1}{2} \frac{\partial g_{\tau}^{\kappa}}{\partial x^v} + x^{\tau} \frac{1}{2} \frac{\partial g_{\tau}^{\mu}}{\partial x^{\tau}} \frac{1}{2} \frac{\partial g_{\tau}^{\nu}}{\partial x^{\tau}} \frac{1}{2} \frac{\partial g_{\tau}^{\kappa}}{\partial x^v}
\end{aligned} \quad (51)$$

as if Binary Law being satisfied for all index except the dummy index of (50). If (51) is a tensor equation, the dummy index of (51) can't make μ or ν . On the other hand, If all coordinate systems satisfies Binary Law, dummy index of (51) should be μ or ν in consideration of Definision5. I aim at the coexistence of these two demands. I rewrite (51) using Definision2 and get

[illegible]

(52) satisfies two demands mentioned above together here. I get

$$x^{\mu;\mu;\mu;\mu} = \frac{\partial^3 x^\mu}{\partial x_\mu \partial x_\mu \partial x_\mu} - \frac{\partial}{\partial x_\mu} \left(\frac{\partial x^\mu}{\partial x^\nu} \frac{1}{2} \frac{\partial g^{\mu\nu}}{\partial x_\mu} \right) - \frac{\partial^2 x^\mu}{\partial x^\nu \partial x_\mu} \frac{1}{2} \frac{\partial g^{\mu\nu}}{\partial x_\mu} - \frac{\partial^2 x^\mu}{\partial x_\mu \partial x^\nu} \frac{1}{2} \frac{\partial g^{\mu\nu}}{\partial x_\mu} + \frac{\partial x^\mu}{\partial x^\nu} \frac{1}{2} \frac{\partial g^{\mu\nu}}{\partial x^\nu} \frac{1}{2} \frac{\partial g^{\mu\nu}}{\partial x_\mu} \quad (53)$$

in consideration of Definision6 for (52). I rewrite (51) using Definision4 and get

[illegible]

(54) satisfies two demands mentioned above together here. I get

$$-x_{;\mu;\mu;\mu}^{\mu} = -\frac{\partial^3 x^{\mu}}{\partial x^{\mu} \partial x^{\mu} \partial x^{\mu}} \quad (55)$$

in consideration of Definition 6 for (54). Because the second term of the right side doesn't exist in (55),

$$x_{;v;v;v}^{\mu} = \frac{\partial^3 x^{\mu}}{\partial x^v \partial x^v \partial x^v} \quad (56)$$

can rewrite (55) using Definision4.

If all coordinate systems satisfies Binary Law in

$$\begin{aligned} & \left\{ \frac{\partial g_{T\epsilon}}{\partial x^v} - \frac{\partial g_{TV}}{\partial x^\epsilon}, \frac{\partial g_{I\epsilon}}{\partial x^\sigma} - \frac{\partial g_{I\sigma}}{\partial x^\epsilon}, \frac{\partial g_{V\epsilon}}{\partial x^\sigma} - \frac{\partial g_{V\sigma}}{\partial x^\epsilon}, \frac{\partial g_{T\epsilon}}{\partial x^l} - \frac{\partial g_{Tl}}{\partial x^\epsilon}, \frac{\partial g_{K\epsilon}}{\partial x^\lambda} - \frac{\partial g_{K\lambda}}{\partial x^\epsilon}, \frac{\partial g_{V\epsilon}}{\partial x^\lambda} - \frac{\partial g_{V\lambda}}{\partial x^\epsilon}, \frac{\partial g_{T\epsilon}}{\partial x^k} - \frac{\partial g_{Tk}}{\partial x^\epsilon} \right\}, \\ & \left\{ \frac{\partial g_{K\epsilon}}{\partial x^\sigma} - \frac{\partial g_{K\sigma}}{\partial x^\epsilon}, \frac{\partial g_{\sigma\epsilon}}{\partial x^\lambda} - \frac{\partial g_{\sigma\lambda}}{\partial x^\epsilon}, \frac{\partial g_{I\epsilon}}{\partial x^k} - \frac{\partial g_{Ik}}{\partial x^\epsilon}, \frac{\partial g_{V\epsilon}}{\partial x^k} - \frac{\partial g_{Vk}}{\partial x^\epsilon} \right\}, \\ & \frac{\partial g_{T\epsilon}}{\partial x^v} - \frac{\partial g_{TV}}{\partial x^\epsilon} = 0, \frac{\partial g_{I\epsilon}}{\partial x^\sigma} - \frac{\partial g_{I\sigma}}{\partial x^\epsilon} = 0, \frac{\partial g_{V\epsilon}}{\partial x^\sigma} - \frac{\partial g_{V\sigma}}{\partial x^\epsilon} = 0, \frac{\partial g_{T\epsilon}}{\partial x^l} - \frac{\partial g_{Tl}}{\partial x^\epsilon} = 0, \frac{\partial g_{K\epsilon}}{\partial x^\lambda} - \frac{\partial g_{K\lambda}}{\partial x^\epsilon} = 0, \frac{\partial g_{V\epsilon}}{\partial x^\lambda} - \frac{\partial g_{V\lambda}}{\partial x^\epsilon} = 0, \\ & \frac{\partial g_{T\epsilon}}{\partial x^k} - \frac{\partial g_{Tk}}{\partial x^\epsilon} = 0, \frac{\partial g_{K\epsilon}}{\partial x^\sigma} - \frac{\partial g_{K\sigma}}{\partial x^\epsilon} = 0, \frac{\partial g_{\sigma\epsilon}}{\partial x^\lambda} - \frac{\partial g_{\sigma\lambda}}{\partial x^\epsilon} = 0, \frac{\partial g_{I\epsilon}}{\partial x^k} - \frac{\partial g_{Ik}}{\partial x^\epsilon} = 0, \frac{\partial g_{V\epsilon}}{\partial x^k} - \frac{\partial g_{Vk}}{\partial x^\epsilon} = 0 \quad (57) \end{aligned}$$

is established. I get

$$x_{;\nu;\sigma;\lambda}^{\mu} = \frac{\partial^3 x^{\mu}}{\partial x^{\nu} \partial x^{\sigma} \partial x^{\lambda}} + \frac{\partial^2}{\partial x^{\sigma} \partial x^{\lambda}} \left(x^{\tau} \frac{1}{2} g^{\epsilon\mu} \left(\frac{\partial g_{\nu\epsilon}}{\partial x^{\tau}} \right) \right) + \frac{\partial}{\partial x^{\lambda}} \left(\frac{\partial x^{\iota}}{\partial x^{\nu}} \frac{1}{2} g^{\epsilon\mu} \left(\frac{\partial g_{\sigma\epsilon}}{\partial x^{\iota}} \right) \right) \\ + \frac{\partial}{\partial x^{\lambda}} \left(x^{\tau} \frac{1}{2} g^{\epsilon\iota} \left(\frac{\partial g_{\nu\epsilon}}{\partial x^{\tau}} \right) \frac{1}{2} g^{\epsilon\mu} \left(\frac{\partial g_{\sigma\epsilon}}{\partial x^{\iota}} \right) \right) - \frac{\partial}{\partial x^{\lambda}} \left(\frac{\partial x^{\mu}}{\partial x^{\iota}} \frac{1}{2} g^{\epsilon\iota} \left(\frac{\partial g_{\sigma\epsilon}}{\partial x^{\nu}} \right) \right) - \frac{\partial}{\partial x^{\lambda}} \left(x^{\tau} \frac{1}{2} g^{\epsilon\mu} \left(\frac{\partial g_{\iota\epsilon}}{\partial x^{\tau}} \right) \frac{1}{2} g^{\epsilon\iota} \left(\frac{\partial g_{\sigma\epsilon}}{\partial x^{\nu}} \right) \right)$$

[illegible]

from (57), Definition 12. I get

$$\begin{aligned}
x_{;v;v;v}^{\mu} &= \frac{\partial^3 x^{\mu}}{\partial x^v \partial x^v \partial x^v} + \frac{\partial^2}{\partial x^v \partial x^v} \left(x^{\tau} \frac{1}{2} \frac{\partial g_{\nu}^{\mu}}{\partial x^{\tau}} \right) + \frac{\partial}{\partial x^v} \left(\frac{\partial x^{\iota}}{\partial x^v} \frac{1}{2} \frac{\partial g_{\nu}^{\mu}}{\partial x^{\iota}} \right) + \frac{\partial}{\partial x^v} \left(x^{\tau} \frac{1}{2} \frac{\partial g_{\nu}^{\iota}}{\partial x^{\tau}} \frac{1}{2} \frac{\partial g_{\nu}^{\mu}}{\partial x^{\iota}} \right) \\
&- \frac{\partial}{\partial x^v} \left(\frac{\partial x^{\mu}}{\partial x^{\iota}} \frac{1}{2} \frac{\partial g_{\nu}^{\iota}}{\partial x^v} \right) - \frac{\partial}{\partial x^v} \left(x^{\tau} \frac{1}{2} \frac{\partial g_{\iota}^{\mu}}{\partial x^{\tau}} \frac{1}{2} \frac{\partial g_{\nu}^{\iota}}{\partial x^v} \right) + \frac{\partial^2 x^{\kappa}}{\partial x^v \partial x^v} \frac{1}{2} \frac{\partial g_{\nu}^{\mu}}{\partial x^{\kappa}} + \frac{\partial}{\partial x^v} \left(x^{\tau} \frac{1}{2} \frac{\partial g_{\nu}^{\kappa}}{\partial x^{\tau}} \frac{1}{2} \frac{\partial g_{\nu}^{\mu}}{\partial x^{\kappa}} \right) \\
&+ \frac{\partial x^{\iota}}{\partial x^v} \frac{1}{2} \frac{\partial g_{\nu}^{\kappa}}{\partial x^{\iota}} \frac{1}{2} \frac{\partial g_{\nu}^{\mu}}{\partial x^{\kappa}} + x^{\tau} \frac{1}{2} \frac{\partial g_{\nu}^{\iota}}{\partial x^{\tau}} \frac{1}{2} \frac{\partial g_{\nu}^{\kappa}}{\partial x^{\iota}} \frac{1}{2} \frac{\partial g_{\nu}^{\mu}}{\partial x^{\kappa}} - \frac{\partial x^{\kappa}}{\partial x^{\iota}} \frac{1}{2} \frac{\partial g_{\nu}^{\iota}}{\partial x^{\kappa}} \frac{1}{2} \frac{\partial g_{\nu}^{\mu}}{\partial x^{\iota}} - x^{\tau} \frac{1}{2} \frac{\partial g_{\nu}^{\kappa}}{\partial x^{\tau}} \frac{1}{2} \frac{\partial g_{\nu}^{\iota}}{\partial x^{\tau}} \frac{1}{2} \frac{\partial g_{\nu}^{\mu}}{\partial x^{\kappa}} \\
&- \frac{\partial^2 x^{\mu}}{\partial x^{\kappa} \partial x^v} \frac{1}{2} \frac{\partial g_{\nu}^{\kappa}}{\partial x^v} - \frac{\partial}{\partial x^v} \left(x^{\tau} \frac{1}{2} \frac{\partial g_{\kappa}^{\mu}}{\partial x^{\tau}} \frac{1}{2} \frac{\partial g_{\nu}^{\kappa}}{\partial x^v} - \frac{\partial x^{\iota}}{\partial x^{\kappa}} \frac{1}{2} \frac{\partial g_{\nu}^{\iota}}{\partial x^{\kappa}} \frac{1}{2} \frac{\partial g_{\nu}^{\kappa}}{\partial x^v} - x^{\tau} \frac{1}{2} \frac{\partial g_{\kappa}^{\iota}}{\partial x^{\tau}} \frac{1}{2} \frac{\partial g_{\nu}^{\iota}}{\partial x^{\kappa}} \frac{1}{2} \frac{\partial g_{\nu}^{\kappa}}{\partial x^v} \right) \\
&+ \frac{\partial x^{\mu}}{\partial x^{\iota}} \frac{1}{2} \frac{\partial g_{\nu}^{\iota}}{\partial x^{\iota}} \frac{1}{2} \frac{\partial g_{\nu}^{\kappa}}{\partial x^v} + x^{\tau} \frac{1}{2} \frac{\partial g_{\iota}^{\mu}}{\partial x^{\tau}} \frac{1}{2} \frac{\partial g_{\nu}^{\iota}}{\partial x^{\tau}} \frac{1}{2} \frac{\partial g_{\nu}^{\kappa}}{\partial x^v} - \frac{\partial^2 x^{\mu}}{\partial x^v \partial x^{\kappa}} \frac{1}{2} \frac{\partial g_{\nu}^{\kappa}}{\partial x^v} - \frac{\partial}{\partial x^{\kappa}} \left(x^{\tau} \frac{1}{2} \frac{\partial g_{\nu}^{\mu}}{\partial x^{\tau}} \frac{1}{2} \frac{\partial g_{\nu}^{\kappa}}{\partial x^v} \right) \\
&- \frac{\partial x^{\iota}}{\partial x^v} \frac{1}{2} \frac{\partial g_{\kappa}^{\mu}}{\partial x^{\iota}} \frac{1}{2} \frac{\partial g_{\nu}^{\kappa}}{\partial x^v} - x^{\tau} \frac{1}{2} \frac{\partial g_{\nu}^{\iota}}{\partial x^{\tau}} \frac{1}{2} \frac{\partial g_{\kappa}^{\iota}}{\partial x^{\tau}} \frac{1}{2} \frac{\partial g_{\nu}^{\kappa}}{\partial x^v} + \frac{\partial x^{\mu}}{\partial x^{\iota}} \frac{1}{2} \frac{\partial g_{\kappa}^{\iota}}{\partial x^{\iota}} \frac{1}{2} \frac{\partial g_{\nu}^{\kappa}}{\partial x^v} + x^{\tau} \frac{1}{2} \frac{\partial g_{\iota}^{\mu}}{\partial x^{\tau}} \frac{1}{2} \frac{\partial g_{\kappa}^{\iota}}{\partial x^{\tau}} \frac{1}{2} \frac{\partial g_{\nu}^{\kappa}}{\partial x^v}
\end{aligned} \tag{59}$$

as if Binary Law being satisfied for all index except the dummy index of (58). If (59) is a tensor equation, the dummy index of (59) can't make μ or ν . On the other hand, If all coordinate systems satisfies Binary Law, dummy index of (59) should be μ or ν in consideration of Definision5. I aim at the coexistence of these two demands. I rewrite (59) using Definision2 and get

$$x^{\mu;\mu;\mu;\mu} = \frac{\partial^3 x^\mu}{\partial x_\mu \partial x_\mu \partial x_\mu} + \frac{\partial^2}{\partial x_\mu \partial x_\mu} \left(x^\tau \frac{1}{2} \frac{\partial g^{\mu\mu}}{\partial x^\tau} \right) + \frac{\partial}{\partial x_\mu} \left(\frac{\partial x^\iota}{\partial x_\mu} \frac{1}{2} \frac{\partial g^{\mu\mu}}{\partial x^\iota} \right) + \frac{\partial}{\partial x_\mu} \left(x^\tau \frac{1}{2} \frac{\partial g^{\mu\iota}}{\partial x^\tau} \frac{1}{2} \frac{\partial g^{\mu\mu}}{\partial x^\iota} \right) \\ - \frac{\partial}{\partial x_{\mu\iota}} \left(\frac{\partial x^\mu}{\partial x^\iota} \frac{1}{2} \frac{\partial g^{\mu\iota}}{\partial x_{\mu\iota}} \right) - \frac{\partial}{\partial x_{\mu\iota}} \left(x^\tau \frac{1}{2} \frac{\partial g^\mu}{\partial x^\tau} \frac{1}{2} \frac{\partial g^{\mu\iota}}{\partial x_{\mu\iota}} \right) + \frac{\partial^2 x^\kappa}{\partial x_{\mu\iota} \partial x_{\mu\iota}} \frac{1}{2} \frac{\partial g^{\mu\mu}}{\partial x^\kappa} + \frac{\partial}{\partial x_{\mu\iota}} \left(x^\tau \frac{1}{2} \frac{\partial g^{\mu\kappa}}{\partial x^\tau} \right) \frac{1}{2} \frac{\partial g^{\mu\mu}}{\partial x^\kappa}$$

[illegible]

(60) satisfies two demands mentioned above together here. I get

$$\begin{aligned}
x^{\mu;\mu;\mu;\mu} = & \frac{\partial^3 x^\mu}{\partial x_\mu \partial x_\mu \partial x_\mu} + \frac{\partial^2}{\partial x_\mu \partial x_\mu} \left(x^\nu \frac{1}{2} \frac{\partial g^{\mu\mu}}{\partial x^\nu} \right) + \frac{\partial}{\partial x_\mu} \left(\frac{\partial x^\nu}{\partial x_\mu} \frac{1}{2} \frac{\partial g^{\mu\mu}}{\partial x^\nu} \right) + \frac{\partial}{\partial x_\mu} \left(x^\nu \frac{1}{2} \frac{\partial g^{\mu\nu}}{\partial x^\nu} \frac{1}{2} \frac{\partial g^{\mu\mu}}{\partial x^\nu} \right) \\
& - \frac{\partial}{\partial x_\mu} \left(\frac{\partial x^\mu}{\partial x^\nu} \frac{1}{2} \frac{\partial g^{\mu\nu}}{\partial x_\mu} \right) + \frac{\partial^2 x^\nu}{\partial x_\mu \partial x_\mu} \frac{1}{2} \frac{\partial g^{\mu\mu}}{\partial x^\nu} + \frac{\partial}{\partial x_\mu} \left(x^\nu \frac{1}{2} \frac{\partial g^{\mu\nu}}{\partial x^\nu} \right) \frac{1}{2} \frac{\partial g^{\mu\mu}}{\partial x^\nu} + \frac{\partial x^\nu}{\partial x_\mu} \frac{1}{2} \frac{\partial g^{\mu\nu}}{\partial x^\nu} \frac{1}{2} \frac{\partial g^{\mu\mu}}{\partial x^\nu} \\
& + x^\nu \frac{1}{2} \frac{\partial g^{\mu\nu}}{\partial x^\nu} \frac{1}{2} \frac{\partial g^{\mu\nu}}{\partial x^\nu} \frac{1}{2} \frac{\partial g^{\mu\mu}}{\partial x^\nu} - \frac{\partial x^\nu}{\partial x^\nu} \frac{1}{2} \frac{\partial g^{\mu\nu}}{\partial x_\mu} \frac{1}{2} \frac{\partial g^{\mu\mu}}{\partial x^\nu} - \frac{\partial^2 x^\mu}{\partial x^\nu \partial x_\mu} \frac{1}{2} \frac{\partial g^{\mu\nu}}{\partial x_\mu} - \frac{\partial x^\nu}{\partial x^\nu} \frac{1}{2} \frac{\partial g^{\mu\mu}}{\partial x^\nu} \frac{1}{2} \frac{\partial g^{\mu\nu}}{\partial x_\mu} \\
& + \frac{\partial x^\mu}{\partial x^\nu} \frac{1}{2} \frac{\partial g^{\mu\nu}}{\partial x^\nu} \frac{1}{2} \frac{\partial g^{\mu\nu}}{\partial x_\mu} - \frac{\partial^2 x^\mu}{\partial x_\mu \partial x^\nu} \frac{1}{2} \frac{\partial g^{\mu\nu}}{\partial x_\mu} - \frac{\partial}{\partial x^\nu} \left(x^\nu \frac{1}{2} \frac{\partial g^{\mu\mu}}{\partial x^\nu} \right) \frac{1}{2} \frac{\partial g^{\mu\nu}}{\partial x_\mu} \quad (61)
\end{aligned}$$

in consideration of Definision6 for (60). I rewrite (59) using Definision4 and get

[illegible]

$$\begin{aligned}
& + \frac{\partial^2 x^\mu}{\partial x^\nu \partial x^\mu} \frac{1}{2} \frac{\partial g_\mu^\nu}{\partial x^\mu} + \frac{\partial}{\partial x^\mu} \left(x^\nu \frac{1}{2} \frac{\partial g_\nu^\mu}{\partial x^\nu} \right) \frac{1}{2} \frac{\partial g_\mu^\nu}{\partial x^\mu} + \frac{\partial x^\nu}{\partial x^\mu} \frac{1}{2} \frac{\partial g_\mu^\nu}{\partial x^\nu} \frac{1}{2} \frac{\partial g_\mu^\nu}{\partial x^\mu} + x^\nu \frac{1}{2} \frac{\partial g_\nu^\mu}{\partial x^\nu} \frac{1}{2} \frac{\partial g_\mu^\nu}{\partial x^\nu} \frac{1}{2} \frac{\partial g_\mu^\nu}{\partial x^\mu} \\
& - \frac{\partial x^\mu}{\partial x^\nu} \frac{1}{2} \frac{\partial g_\nu^\mu}{\partial x^\nu} \frac{1}{2} \frac{\partial g_\mu^\nu}{\partial x^\mu} - x^\nu \frac{1}{2} \frac{\partial g_\nu^\mu}{\partial x^\nu} \frac{1}{2} \frac{\partial g_\mu^\nu}{\partial x^\nu} \frac{1}{2} \frac{\partial g_\mu^\nu}{\partial x^\mu} + \frac{\partial^2 x^\mu}{\partial x^\mu \partial x^\nu} \frac{1}{2} \frac{\partial g_\mu^\nu}{\partial x^\mu} + \frac{\partial}{\partial x^\nu} \left(x^\nu \frac{1}{2} \frac{\partial g_\mu^\mu}{\partial x^\mu} \right) \frac{1}{2} \frac{\partial g_\mu^\nu}{\partial x^\mu} \\
& + \frac{\partial x^\nu}{\partial x^\mu} \frac{1}{2} \frac{\partial g_\nu^\mu}{\partial x^\nu} \frac{1}{2} \frac{\partial g_\mu^\nu}{\partial x^\mu} + x^\nu \frac{1}{2} \frac{\partial g_\nu^\mu}{\partial x^\nu} \frac{1}{2} \frac{\partial g_\mu^\nu}{\partial x^\nu} \frac{1}{2} \frac{\partial g_\mu^\nu}{\partial x^\mu} - \frac{\partial x^\mu}{\partial x^\nu} \frac{1}{2} \frac{\partial g_\nu^\mu}{\partial x^\nu} \frac{1}{2} \frac{\partial g_\mu^\nu}{\partial x^\mu} - x^\nu \frac{1}{2} \frac{\partial g_\nu^\mu}{\partial x^\nu} \frac{1}{2} \frac{\partial g_\mu^\nu}{\partial x^\nu} \frac{1}{2} \frac{\partial g_\mu^\nu}{\partial x^\mu}.
\end{aligned} \quad (62)$$

(62) satisfies two demands mentioned above together here. I get

$$-x_{;\mu;\mu;\mu}^\mu = -\frac{\partial^3 x^\mu}{\partial x^\mu \partial x^\mu \partial x^\mu} \quad (63)$$

in consideration of Definision6 for (62). Because the second term of the right side doesn't exist in (63), (56) can rewrite (63) using Definision4. End Proof

ABOUT A COORDINATE TRANSFORMATIONS EQUATION IN TENSOR SATISFYING BINARY LAW

Proposition6 When all coordinate systems satisfies Binary Law, $x_\mu^\mu = x_\nu^\nu$ is established for x_μ^μ components of a tensor satisfying Binary law of rank zero.

Proof: If all coordinate systems satisfies Binary Law, I get

$$x_\mu^\mu = \frac{\partial x^\mu}{\partial x^\nu} \frac{\partial x^\nu}{\partial x^\mu} x_\nu^\nu \quad (64)$$

from Definision13. Because (64) accords in Definision13, the components of a tensor of rank zero is equivalent with components of a tensor satisfying Binary law of rank zero. If all coordinate systems satisfies Binary Law, I get

$$x^\mu = \frac{\partial x^\mu}{\partial x^\nu} x^\nu \quad (65)$$

from Definision14. Because (65) accords in Definision14, the contravariant components of a tensor of the first rank is equivalent with contravariant components of a tensor satisfying Binary law of the first rank. If all coordinate systems satisfies Binary Law, I get

$$x_\mu = \frac{\partial x^\nu}{\partial x^\mu} x_\nu \quad (66)$$

from Definision15. Because (66) accords in Definision15, the covariant components of a tensor of the first rank is equivalent with covariant components of a tensor satisfying Binary law of the first rank. End Proof

Proposition7 When all coordinate systems satisfies Binary Law, $x^{\mu\nu} = x^{\nu\mu}$ is established for $x^{\mu\nu}$ contravariant components of a tensor satisfying Binary law of the second rank.

Proof: I get

$$x^{\mu\nu} = \frac{\partial x^\mu}{\partial x^\sigma} \frac{\partial x^\nu}{\partial x^\lambda} x^{\sigma\lambda} \quad (67)$$

as if Binary Law being satisfied for all index except the dummy index of Definision16. If all coordinate systems satisfies Binary Law, dummy index of (67) should be μ or ν in consideration of Definision5. Thus, I rewrite dummy index of (67) and get

$$x^{\mu\nu} = \frac{\partial x^\mu}{\partial x^\mu} \frac{\partial x^\nu}{\partial x^\nu} x^{\mu\nu} = x^{\mu\nu}, \quad (68)$$

$$x^{\mu\nu} = \frac{\partial x^\mu}{\partial x^\nu} \frac{\partial x^\nu}{\partial x^\nu} x^{\nu\nu} = \frac{\partial x^\mu}{\partial x^\nu} x^{\nu\nu}, \quad (69)$$

$$x^{\mu\nu} = \frac{\partial x^\mu}{\partial x^\nu} \frac{\partial x^\nu}{\partial x^\mu} x^{\nu\mu}, \quad (70)$$

$$x^{\mu\nu} = \frac{\partial x^\mu}{\partial x^\mu} \frac{\partial x^\nu}{\partial x^\mu} x^{\mu\mu} = \frac{\partial x^\nu}{\partial x^\mu} x^{\mu\mu}. \quad (71)$$

I decide not to handle (68),(69),(71) because $\frac{\partial x^\mu}{\partial x^\mu}, \frac{\partial x^\nu}{\partial x^\nu}$ exists in (68),(69),(71). I rewrite (70) using Definision2,Definision3 and get

$$x_\mu^\mu = \frac{\partial x^\mu}{\partial x^\nu} \frac{\partial x_\mu}{\partial x_\nu} x_\nu^\nu. \quad (72)$$

I rewrite $x^{\mu\nu} = x^{\nu\mu}, x^{\nu\mu} = x^{\nu\mu}$ using Definision2,Definision3 and get

$$x^{\mu\nu} = x_\mu^\mu, x^{\nu\mu} = x_\nu^\nu. \quad (73)$$

I get

$$\begin{aligned} \frac{\partial x^\mu}{\partial x^\nu} \frac{\partial x^\nu}{\partial x^\mu} x^{\nu\mu} &= \frac{\partial x^\mu}{\partial x^\nu} \frac{\partial x_\mu}{\partial x_\nu} x_\nu^\nu \\ &= \frac{\partial x^\mu}{\partial x^\nu} \frac{\partial x_\mu}{\partial x_\nu} x^{\nu\mu} \end{aligned} \quad (74)$$

from (70),(72),(73). I get

$$x^{\mu\nu} = x_\mu^\mu = x_\nu^\nu = x^{\nu\mu} \quad (75)$$

from (72),(73). End Proof

Proposition8 When all coordinate systems satisfies Binary Law, $x_{\mu\nu} = x_{\nu\mu}$ is established for $x_{\mu\nu}$ covariant components of a tensor satisfying Binary law of the second rank.

Proof: I get

$$x_{\mu\nu} = \frac{\partial x^\sigma}{\partial x^\mu} \frac{\partial x^\lambda}{\partial x^\nu} x_{\sigma\lambda} \quad (76)$$

as if Binary Law being satisfied for all index except the dummy index of Definision17. If all coordinate systems satisfies Binary Law, dummy index of (76) should be μ or ν in consideration of Definision5. Thus, I rewrite dummy index of (76) and get

$$x_{\mu\nu} = \frac{\partial x^\mu}{\partial x^\mu} \frac{\partial x^\nu}{\partial x^\nu} x_{\mu\nu} = x_{\mu\nu}, \quad (77)$$

$$x_{\mu\nu} = \frac{\partial x^\mu}{\partial x^\mu} \frac{\partial x^\nu}{\partial x^\nu} x_{\nu\nu} = \frac{\partial x^\nu}{\partial x^\mu} x_{\nu\nu}, \quad (78)$$

$$x_{\mu\nu} = \frac{\partial x^\nu}{\partial x^\mu} \frac{\partial x^\mu}{\partial x^\nu} x_{\nu\mu}, \quad (79)$$

$$x_{\mu\nu} = \frac{\partial x^\mu}{\partial x^\mu} \frac{\partial x^\mu}{\partial x^\nu} x_{\mu\mu} = \frac{\partial x^\mu}{\partial x^\nu} x_{\mu\mu}. \quad (80)$$

I decide not to handle (77),(78),(80) because $\frac{\partial x^\mu}{\partial x^\mu}, \frac{\partial x^\nu}{\partial x^\nu}$ exists in (77),(78),(80). I rewrite (79) using Definision2,Definision3 and get

$$x_\nu^\nu = \frac{\partial x_\mu}{\partial x_\nu} \frac{\partial x^\mu}{\partial x^\nu} x_\mu^\mu. \quad (81)$$

I rewrite $x_{\mu\nu} = x_{\mu\nu}, x_{\nu\mu} = x_{\nu\mu}$ using Definision2,Definision3 and get

$$x_{\mu\nu} = x_\nu^\nu, x_{\nu\mu} = x_\mu^\mu. \quad (82)$$

I get

$$\frac{\partial x^\nu}{\partial x^\mu} \frac{\partial x^\mu}{\partial x^\nu} x_{\nu\mu} = \frac{\partial x_\mu}{\partial x_\nu} \frac{\partial x^\mu}{\partial x^\nu} x_\mu^\mu$$

$$= \frac{\partial x_\mu}{\partial x_\nu} \frac{\partial x^\mu}{\partial x^\nu} x_{\nu\mu} \quad (83)$$

from (79),(81),(82). I get

$$x_{\mu\nu} = x_\nu^\nu = x_\mu^\mu = x_{\nu\mu} \quad (84)$$

from (81),(82). End Proof

Proposition9 When all coordinate systems satisfies Binary Law, $x_\nu^\mu = \frac{\partial x^\mu}{\partial x^\nu} \frac{\partial x^\mu}{\partial x^\nu} x_\mu^\nu$ is established for x_ν^μ components of the mixed tensor satisfying Binary law of the second rank.

Proof: I get

$$x_\nu^\mu = \frac{\partial x^\mu}{\partial x^\sigma} \frac{\partial x^\lambda}{\partial x^\nu} x_\lambda^\sigma \quad (85)$$

as if Binary Law being satisfied for all index except the dummy index of Definision18. If all coordinate systems satisfies Binary Law, dummy index of (85) should be μ or ν in consideration of Definision5. Thus, I rewrite dummy index of (85) and get

$$x_\nu^\mu = \frac{\partial x^\mu}{\partial x^\mu} \frac{\partial x^\nu}{\partial x^\nu} x_\nu^\mu = x_\nu^\mu, \quad (86)$$

$$x_\nu^\mu = \frac{\partial x^\mu}{\partial x^\nu} \frac{\partial x^\nu}{\partial x^\nu} x_\nu^\nu = \frac{\partial x^\mu}{\partial x^\nu} x_\nu^\nu, \quad (87)$$

$$x_\nu^\mu = \frac{\partial x^\mu}{\partial x^\nu} \frac{\partial x^\mu}{\partial x^\nu} x_\mu^\nu, \quad (88)$$

$$x_\nu^\mu = \frac{\partial x^\mu}{\partial x^\mu} \frac{\partial x^\mu}{\partial x^\nu} x_\mu^\mu = \frac{\partial x^\mu}{\partial x^\nu} x_\mu^\mu. \quad (89)$$

I decide not to handle (86),(87),(89) because $\frac{\partial x^\mu}{\partial x^\mu}, \frac{\partial x^\nu}{\partial x^\nu}$ exists in (86),(87),(89). End Proof

Proposition10 When all coordinate systems satisfies Binary Law, $x_{\mu\nu\nu} = \frac{\partial x^\nu}{\partial x^\mu} \frac{\partial x^\mu}{\partial x^\nu} \frac{\partial x^\mu}{\partial x^\nu} x_{\nu\mu\mu}$ is established for $x_{\mu\nu\nu}$ covariant components of a tensor satisfying Binary law of the third rank.

Proof: I get

$$x_{\mu\nu\nu} = \frac{\partial x^\lambda}{\partial x^\mu} \frac{\partial x^\iota}{\partial x^\nu} \frac{\partial x^\epsilon}{\partial x^\nu} x_{\lambda\iota\epsilon} \quad (90)$$

as if Binary Law being satisfied for all index except the dummy index of Definision19. If all coordinate systems satisfies Binary Law, dummy index of (90) should be μ or ν in consideration of Definision5. Thus, I rewrite dummy index of (90) and get

$$x_{\mu\nu\nu} = \frac{\partial x^\mu}{\partial x^\mu} \frac{\partial x^\nu}{\partial x^\nu} \frac{\partial x^\nu}{\partial x^\nu} x_{\mu\nu\nu} = x_{\mu\nu\nu}, \quad (91)$$

$$x_{\mu\nu\nu} = \frac{\partial x^\nu}{\partial x^\mu} \frac{\partial x^\nu}{\partial x^\nu} \frac{\partial x^\nu}{\partial x^\nu} x_{\nu\nu\nu} = \frac{\partial x^\nu}{\partial x^\mu} x_{\nu\nu\nu}, \quad (92)$$

$$x_{\mu\nu\nu} = \frac{\partial x^\nu}{\partial x^\nu} \frac{\partial x^\mu}{\partial x^\nu} \frac{\partial x^\nu}{\partial x^\nu} x_{\nu\mu\nu} = \frac{\partial x^\nu}{\partial x^\mu} \frac{\partial x^\mu}{\partial x^\nu} x_{\nu\mu\nu}, \quad (93)$$

$$x_{\mu\nu\nu} = \frac{\partial x^\nu}{\partial x^\mu} \frac{\partial x^\mu}{\partial x^\nu} \frac{\partial x^\mu}{\partial x^\nu} x_{\nu\mu\mu}, \quad (94)$$

$$x_{\mu\nu\nu} = \frac{\partial x^\mu}{\partial x^\mu} \frac{\partial x^\nu}{\partial x^\nu} \frac{\partial x^\mu}{\partial x^\nu} x_{\mu\mu\mu} = \frac{\partial x^\mu}{\partial x^\nu} \frac{\partial x^\mu}{\partial x^\nu} x_{\mu\mu\mu}, \quad (95)$$

$$x_{\mu\nu\nu} = \frac{\partial x^\mu}{\partial x^\mu} \frac{\partial x^\nu}{\partial x^\nu} \frac{\partial x^\mu}{\partial x^\nu} x_{\mu\nu\mu} = \frac{\partial x^\mu}{\partial x^\nu} x_{\mu\nu\mu}. \quad (96)$$

I decide not to handle (91),(92),(93),(95),(96) because $\frac{\partial x^\mu}{\partial x^\mu}, \frac{\partial x^\nu}{\partial x^\nu}$ exists in (91),(92),(93),(95),(96). Because two same index is existing in $x_{\mu\nu\nu}, x_{\nu\mu\mu}$ of (94), it is a problem. I rewrite (94) using Definision2,Definision3 and get

$$x_{vv}^v = \frac{\partial x_\mu}{\partial x_v} \frac{\partial x^\mu}{\partial x^v} \frac{\partial x^\mu}{\partial x^v} x_{\mu\mu}^\mu \quad (97)$$

Two same index is existing in $x_{vv}^v, x_{\mu\mu}^\mu$ of (97), but the problem doesn't occur because one is dummy index. Because (97) could rewrite (94), the problem of (94) was solved. I rewrite $x_{\mu vv} = x_{\mu vv}, x_{v\mu\mu} = x_{v\mu\mu}$ using Definision2, Definision3 and get

$$x_{\mu vv} = x_{vv}^v, x_{v\mu\mu} = x_{\mu\mu}^\mu \quad (98)$$

I get

$$\begin{aligned} \frac{\partial x^v}{\partial x^\mu} \frac{\partial x^\mu}{\partial x^v} \frac{\partial x^\mu}{\partial x^v} x_{v\mu\mu} &= \frac{\partial x_\mu}{\partial x_v} \frac{\partial x^\mu}{\partial x^v} \frac{\partial x^\mu}{\partial x^v} x_{\mu\mu}^\mu \\ &= \frac{\partial x_\mu}{\partial x_v} \frac{\partial x^\mu}{\partial x^v} \frac{\partial x^\mu}{\partial x^v} x_{v\mu\mu} \end{aligned} \quad (99)$$

from (94),(97),(98). End Proof

Proposition11 When all coordinate systems satisfies Binary Law, $x_{vv}^\mu = \frac{\partial x^v}{\partial x^\mu} x_{\mu\mu}^v$ is established for x_{vv}^μ components of the mixed tensor satisfying Binary law of the third rank of the second rank covariant in the first rank contravariant.

Proof: I get

$$x_{vv}^\mu = \frac{\partial x^\mu}{\partial x^\lambda} \frac{\partial x^\lambda}{\partial x^v} \frac{\partial x^\epsilon}{\partial x^v} x_{\epsilon\epsilon}^\lambda \quad (100)$$

as if Binary Law being satisfied for all index except the dummy index of Definision20. If all coordinate systems satisfies Binary Law, dummy index of (100) should be μ or v in consideration of Definision5. Thus, I rewrite dummy index of (100) and get

$$x_{vv}^\mu = \frac{\partial x^\mu}{\partial x^\mu} \frac{\partial x^v}{\partial x^v} \frac{\partial x^v}{\partial x^v} x_{vv}^\mu = x_{vv}^\mu, \quad (101)$$

$$x_{vv}^\mu = \frac{\partial x^\mu}{\partial x^v} \frac{\partial x^v}{\partial x^v} \frac{\partial x^v}{\partial x^v} x_{vv}^v = \frac{\partial x^\mu}{\partial x^v} x_{vv}^v, \quad (102)$$

$$x_{vv}^\mu = \frac{\partial x^\mu}{\partial x^v} \frac{\partial x^\mu}{\partial x^v} \frac{\partial x^v}{\partial x^v} x_{\mu v}^v = \frac{\partial x^\mu}{\partial x^v} \frac{\partial x^\mu}{\partial x^v} x_{\mu v}^v, \quad (103)$$

$$x_{vv}^\mu = \frac{\partial x^\mu}{\partial x^v} \frac{\partial x^\mu}{\partial x^v} \frac{\partial x^\mu}{\partial x^v} x_{\mu\mu}^v, \quad (104)$$

$$x_{vv}^\mu = \frac{\partial x^\mu}{\partial x^\mu} \frac{\partial x^\mu}{\partial x^v} \frac{\partial x^\mu}{\partial x^v} x_{\mu\mu}^\mu = \frac{\partial x^\mu}{\partial x^v} \frac{\partial x^\mu}{\partial x^v} x_{\mu\mu}^\mu, \quad (105)$$

$$x_{vv}^\mu = \frac{\partial x^\mu}{\partial x^\mu} \frac{\partial x^v}{\partial x^v} \frac{\partial x^\mu}{\partial x^v} x_{v\mu}^\mu = \frac{\partial x^\mu}{\partial x^v} x_{v\mu}^\mu. \quad (106)$$

I decide not to handle (101),(102),(103),(105),(106) because $\frac{\partial x^\mu}{\partial x^\mu}, \frac{\partial x^v}{\partial x^v}$ exists in (101),(102),(103),(105),(106). Because two same index is existing in $x_{vv}^\mu, x_{\mu\mu}^v$ of (104), it is a problem. I rewrite (104) using Definision4 and get

$$x_{\mu\mu}^\mu = \frac{\partial x^\mu}{\partial x^v} \frac{\partial x^v}{\partial x^\mu} \frac{\partial x^v}{\partial x^\mu} x_{vv}^v. \quad (107)$$

Two same index is existing in $x_{\mu\mu}^\mu, x_{vv}^v$ of (107), but the problem doesn't occur because one is dummy index. Because (107) could rewrite (104), the problem of (104) was solved. I rewrite $x_{vv}^\mu = x_{vv}^\mu, x_{\mu\mu}^v = x_{\mu\mu}^v$ using Definision4 and get

$$x_{vv}^\mu = x_{\mu\mu}^\mu, x_{\mu\mu}^v = x_{vv}^v. \quad (108)$$

I get

$$\begin{aligned} \frac{\partial x^\mu}{\partial x^\nu} \frac{\partial x^\mu}{\partial x^\nu} \frac{\partial x^\mu}{\partial x^\nu} x_{\mu\mu}^\nu &= \frac{\partial x^\mu}{\partial x^\nu} \frac{\partial x^\nu}{\partial x^\mu} \frac{\partial x^\nu}{\partial x^\mu} x_{\nu\nu}^\mu \\ &= \frac{\partial x^\mu}{\partial x^\nu} \frac{\partial x^\nu}{\partial x^\mu} \frac{\partial x^\nu}{\partial x^\nu} x_{\mu\mu}^\nu \end{aligned} \quad (109)$$

from (104),(107),(108). I get

$$\frac{\partial x^\mu}{\partial x^\nu} \frac{\partial x^\mu}{\partial x^\nu} = \frac{\partial x^\nu}{\partial x^\mu} \frac{\partial x^\nu}{\partial x^\mu} \quad (110)$$

from (109). End Proof

Proposition12 When all coordinate systems satisfies Binary Law, $x_{\nu\nu\nu}^\mu = x_{\mu\mu\mu}^\nu$ is established for $x_{\nu\nu\nu}^\mu$ components of the mixed tensor satisfying Binary law of the fourth rank of the third rank covariant in the first rank contravariant.

Proof: I get

$$x_{\nu\nu\nu}^\mu = \frac{\partial x^\mu}{\partial x^\epsilon} \frac{\partial x^\epsilon}{\partial x^\nu} \frac{\partial x^\alpha}{\partial x^\nu} \frac{\partial x^\beta}{\partial x^\nu} x_{\epsilon\alpha\beta}^\mu \quad (111)$$

as if Binary Law being satisfied for all index except the dummy index of Definision21. If all coordinate systems satisfies Binary Law, dummy index of (111) should be μ or ν in consideration of Definision5. Thus, I rewrite dummy index of (111) and get

$$x_{\nu\nu\nu}^\mu = \frac{\partial x^\mu}{\partial x^\mu} \frac{\partial x^\nu}{\partial x^\nu} \frac{\partial x^\nu}{\partial x^\nu} \frac{\partial x^\nu}{\partial x^\nu} x_{\nu\nu\nu}^\mu = x_{\nu\nu\nu}^\mu, \quad (112)$$

$$x_{\nu\nu\nu}^\mu = \frac{\partial x^\mu}{\partial x^\nu} \frac{\partial x^\nu}{\partial x^\nu} \frac{\partial x^\nu}{\partial x^\nu} \frac{\partial x^\nu}{\partial x^\nu} x_{\nu\nu\nu}^\mu = \frac{\partial x^\mu}{\partial x^\nu} x_{\nu\nu\nu}^\nu, \quad (113)$$

$$x_{\nu\nu\nu}^\mu = \frac{\partial x^\mu}{\partial x^\nu} \frac{\partial x^\nu}{\partial x^\nu} \frac{\partial x^\nu}{\partial x^\nu} \frac{\partial x^\nu}{\partial x^\nu} x_{\mu\nu\nu}^\nu = \frac{\partial x^\mu}{\partial x^\nu} \frac{\partial x^\mu}{\partial x^\nu} x_{\mu\nu\nu}^\nu, \quad (114)$$

$$x_{\nu\nu\nu}^\mu = \frac{\partial x^\mu}{\partial x^\nu} \frac{\partial x^\nu}{\partial x^\mu} \frac{\partial x^\mu}{\partial x^\nu} \frac{\partial x^\mu}{\partial x^\nu} x_{\mu\mu\nu}^\nu = \frac{\partial x^\mu}{\partial x^\nu} \frac{\partial x^\mu}{\partial x^\nu} \frac{\partial x^\mu}{\partial x^\nu} x_{\mu\mu\nu}^\nu, \quad (115)$$

$$x_{\nu\nu\nu}^\mu = \frac{\partial x^\mu}{\partial x^\mu} \frac{\partial x^\nu}{\partial x^\mu} \frac{\partial x^\mu}{\partial x^\mu} \frac{\partial x^\mu}{\partial x^\mu} x_{\mu\mu\mu}^\nu, \quad (116)$$

$$x_{\nu\nu\nu}^\mu = \frac{\partial x^\mu}{\partial x^\mu} \frac{\partial x^\mu}{\partial x^\nu} \frac{\partial x^\mu}{\partial x^\nu} \frac{\partial x^\mu}{\partial x^\nu} x_{\mu\mu\mu}^\mu = \frac{\partial x^\mu}{\partial x^\nu} \frac{\partial x^\mu}{\partial x^\nu} \frac{\partial x^\mu}{\partial x^\nu} x_{\mu\mu\mu}^\mu, \quad (117)$$

$$x_{\nu\nu\nu}^\mu = \frac{\partial x^\mu}{\partial x^\mu} \frac{\partial x^\nu}{\partial x^\mu} \frac{\partial x^\mu}{\partial x^\nu} \frac{\partial x^\mu}{\partial x^\nu} x_{\nu\mu\mu}^\mu = \frac{\partial x^\mu}{\partial x^\nu} \frac{\partial x^\mu}{\partial x^\nu} x_{\nu\mu\mu}^\mu, \quad (118)$$

$$x_{\nu\nu\nu}^\mu = \frac{\partial x^\mu}{\partial x^\mu} \frac{\partial x^\nu}{\partial x^\nu} \frac{\partial x^\nu}{\partial x^\nu} \frac{\partial x^\mu}{\partial x^\nu} x_{\nu\nu\mu}^\mu = \frac{\partial x^\mu}{\partial x^\nu} x_{\nu\nu\mu}^\mu. \quad (119)$$

I decide not to handle (112),(113),(114),(115),(117),(118),(119) because $\frac{\partial x^\mu}{\partial x^\mu}, \frac{\partial x^\nu}{\partial x^\nu}$ exists in (112),(113),(114),(115),(117),(118),(119). Because three same index is existing in $x_{\nu\nu\nu}^\mu, x_{\mu\mu\mu}^\nu$ of (116), it is a problem. I rewrite (116) using Definision4 and get

$$x_{\mu\mu\nu}^\mu = \frac{\partial x^\mu}{\partial x^\nu} \frac{\partial x^\nu}{\partial x^\mu} \frac{\partial x^\nu}{\partial x^\mu} \frac{\partial x^\mu}{\partial x^\nu} x_{\nu\nu\mu}^\nu. \quad (120)$$

Two same index is existing in $x_{\mu\mu\nu}^\mu, x_{\nu\nu\mu}^\nu$ of (120), but the problem doesn't occur because one is dummy index. Because (120) could rewrite (116), the problem of (116) was solved. I rewrite

$$\begin{aligned} x_{\nu\nu\nu}^\mu &= x_{\nu\nu\nu}^\mu, x_{\mu\mu\mu}^\nu = x_{\mu\mu\mu}^\nu \text{ using Definision4 and get} \\ x_{\nu\nu\nu}^\mu &= x_{\mu\mu\nu}^\mu, x_{\mu\mu\mu}^\nu = x_{\nu\nu\mu}^\nu. \end{aligned} \quad (121)$$

I get

$$\frac{\partial x^\mu}{\partial x^\nu} \frac{\partial x^\mu}{\partial x^\nu} \frac{\partial x^\mu}{\partial x^\nu} x_{\mu\mu\mu}^\nu = \frac{\partial x^\mu}{\partial x^\nu} \frac{\partial x^\nu}{\partial x^\mu} \frac{\partial x^\nu}{\partial x^\mu} \frac{\partial x^\mu}{\partial x^\nu} x_{\nu\nu\mu}^\nu$$

$$= \frac{\partial x^\mu}{\partial x^\nu} \frac{\partial x^\nu}{\partial x^\mu} \frac{\partial x^\nu}{\partial x^\mu} \frac{\partial x^\mu}{\partial x^\nu} x_{\mu\mu\mu}^\nu \quad (122)$$

from (116),(120),(121). I get

$$\frac{\partial x^\mu}{\partial x^\nu} \frac{\partial x^\mu}{\partial x^\nu} = \frac{\partial x^\nu}{\partial x^\mu} \frac{\partial x^\nu}{\partial x^\mu} \quad (123)$$

from (122). I get

$$\begin{aligned} x_{\nu\nu\nu}^\mu &= x_{\mu\mu\nu}^\mu = x_{\nu\nu\mu}^\nu = x_{\mu\mu\mu}^\nu, \\ x_{\nu\nu\nu}^\mu &= x_{\mu\nu}^\mu = x_{\nu\mu}^\nu = x_{\mu\mu\mu}^\nu \end{aligned} \quad (124)$$

from (120),(121). I get

$$x_{\nu\nu\nu}^\mu = x_{\mu\nu}^\mu = x_\nu^\nu = x_\mu^\mu = x_{\nu\mu}^\nu = x_{\mu\mu\mu}^\nu \quad (125)$$

from (84),(124). I rewrite (125) by consideration of $x_{\nu\nu\nu}^\mu \rightarrow x_{;\nu;\nu;\nu}^\mu$, $x_{\mu\mu\mu}^\nu \rightarrow x_{;\mu;\mu;\mu}^\nu$, $x_{\mu\nu}^\nu \rightarrow x_{\mu;\nu}^\nu$, $x_{\nu\mu}^\mu \rightarrow x_{\nu;\mu}^\mu$, $x_\nu^\nu \rightarrow x_{;\nu}^\nu$, $x_\mu^\mu \rightarrow x_{;\mu}^\mu$, (4),(7),(56), μ, ν -inversion form of (4),(7),(56) and get

$$\begin{aligned} \frac{\partial^3 x^\mu}{\partial x^\nu \partial x^\nu \partial x^\nu} &= \frac{\partial x_\mu}{\partial x^\nu} = \frac{\partial x_\nu}{\partial x^\mu} - x_\mu \frac{1}{2} \left(\frac{\partial g^{\nu\mu}}{\partial x^\nu} \right) \\ &= \frac{\partial x_\mu}{\partial x^\mu} - x_\nu \frac{1}{2} \left(\frac{\partial g^{\mu\nu}}{\partial x^\mu} \right) = \frac{\partial x_\nu}{\partial x^\mu} = \frac{\partial^3 x^\nu}{\partial x^\mu \partial x^\mu \partial x^\mu}. \end{aligned} \quad (126)$$

I get

$$\begin{aligned} \frac{\partial^3 x^\mu}{\partial x^\nu \partial x^\nu \partial x^\nu} &= \frac{\partial x_\mu}{\partial x^\nu} = \frac{\partial x_\nu}{\partial x^\mu} - x_\mu \frac{1}{2} \left(\frac{\partial g^{\nu\mu}}{\partial x^\nu} \right) \\ &= \frac{\partial x_\mu}{\partial x^\mu} - x_\nu \frac{1}{2} \left(\frac{\partial g^{\mu\nu}}{\partial x^\mu} \right) = \frac{\partial x_\nu}{\partial x^\mu} = \frac{\partial^3 x^\nu}{\partial x^\mu \partial x^\mu \partial x^\mu} = M \end{aligned} \quad (127)$$

from (126),Definision7. End Proof

DISCUSSION

About Proposition1

If all coordinate systems satisfies Binary Law, I get

$$\begin{aligned} x_{\mu;\nu} &= \frac{\partial x_\mu}{\partial x^\nu} - x_\nu \frac{1}{2} g^{\nu\nu} \left(\frac{\partial g_{\mu\nu}}{\partial x^\nu} + \frac{\partial g_{\nu\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\nu} \right) \\ &= \frac{\partial x_\mu}{\partial x^\nu} - x_\nu \frac{1}{2} g^{\nu\nu} \left(\frac{\partial g_{\nu\nu}}{\partial x^\mu} \right) = \frac{\partial x_\mu}{\partial x^\nu} - x_\nu \frac{1}{2} \frac{\partial g_\nu^\nu}{\partial x^\mu} \end{aligned} \quad (128)$$

from Definision8. At first I applied Binary Law only for a part to show in

$$x_{\mu;\nu} = \frac{\partial x_\mu}{\partial x^\nu} - x_\tau \frac{1}{2} g^{\tau\tau} \left(\frac{\partial g_{\mu\tau}}{\partial x^\nu} + \frac{\partial g_{\nu\tau}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\tau} \right) \quad (129)$$

for Definision8 and got (2) in this article. Then, I applied Binary Law only for a part to show in

$$x_{\hat{\mu};\hat{\nu}} = \frac{\partial x_{\hat{\mu}}}{\partial x^{\hat{\nu}}} - x_{\hat{\tau}} \frac{1}{2} \frac{\partial g_{\hat{\nu}}^{\hat{\tau}}}{\partial x^{\hat{\mu}}} \quad (130)$$

for (2) and got (3). I get (128) as if Binary Law being satisfied for all index of (3).

(128) isn't tensor satisfying Binary Law here. On the other hand, I rewrite (3) using Definision2 and get (4). (4) is tensor satisfying Binary Law here.

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