

Topological Conjugacy and Symbolic Dynamics of the One Dimensional Map

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ABSTRACT

In the study of nonlinear physical systems, one encounters possibly random or chaotic behavior, although the techniques may be perfectly deterministic. Topological conjugacy is essential in the study of iterated functions and, more generally, dynamical systems. Our goal is to show that conjugacy (topological) and symbolic dynamics are a distinguished combination of tools of dynamical systems. This paper investigates conjugacy between different one-dimensional maps and discusses semi-conjugacy between doubling maps and shift maps. Finally, we discuss employing techniques from symbolic dynamics to one-dimensional maps, which means how extended dynamics work for one dimensional map.

Keywords: Symbolic dynamics, Symbol space, Shift map, Cantor set, Topological conjugacy, semi-conjugacy.

INTRODUCTION

The Topological conjugacy feature has an essential role in studying the chaotic behavior of a map. With the help of this feature, we can explore the chaotic significance by comparing one map with another map. Topological conjugacy has such importance as it can protect many topological dynamical properties. In the discussion of any chaotic dynamical system, topological conjugacy between maps is a very powerful tool. Topological conjugacy is a significant concept in the knowledge of dynamical systems. It is an essential tool that makes predictions about a dynamical system's behavior by comparing it with another dynamical system whose specific properties are known. We get the basic idea about symbolic dynamics from many papers and books. There are some interesting applications of a symbolic dynamical system. Such as the symbolic representation of the Cantor set. From the article of H.R. Biswas and Monirul Islam [3], we get the basic idea of shift maps and their chaotic properties. Ju H., Shao H., Choe Y., and Shi Y. [11] narrated conditions for maps to be conjugate (topologically) or semi-conjugate to subshifts of finite type and inference of chaos.

The symbolic dynamical systems form a significant category of dynamical systems. In such a place, we may consider a shift operator that shifts a sequence one symbol left. J. Hadamard [1] gave the example of the first application of symbolic dynamics. He used trajectory coding for the representation of the universal behavior of geodesics on surfaces with negative curvature.

C. Hsu enlarged the cell-to-cell mapping method. Brin [7] discussed cells of a given partition and their images under the action of a system. G. Osipenko [13] introduced the method of a symbolic image in which way an oriented graph represents the transitions of a trajectory of partition elements. Its adjacency matrix defines the Topological Markov chain, and symbol sequences correspond to paths on the graph. This method is effectively applied to the cost of production of invariant sets and Morse spectrum. Symbolic dynamics attempts to answer how much actuality about a dynamical system can be drawn from a data sequence produced by measurements of the system. Symbolic dynamics gives a method for converting real system trajectories into symbol sequences and then answering how much about the underlying system can be deduced from these sequences.

Symbolic dynamics hope that one may convert a dynamical system into an extended technique, study the simplified dynamics in sequences space, and then take the results back to the state-space of the original system. Moreover, there are several works on σ symbolic dynamics where dynamics are represented by maps on symbol spaces.

In Section 2, we describe the mathematical preliminaries, which are requirements for the subsequent chapters. In Section 3, we have proved that $\exists$ a conjugacy for shift map and tent map. Also, in this section, we observed that a conjugacy between doubling map and shift map. In Section 4, we have discussed symbolic dynamics for one dimensional map.

MATHEMATICAL PRELIMINARIES

The symbol space we use here has a metric structure, which is defined in a natural way. So the study can be related to the metric space with the advantage that we can use the standard concepts of dynamical systems available therein. Symbolic dynamical systems are suitable for exalted generalization and abstraction of the original dynamical systems based on the topological conjugacy between the continuous evolution of the dynamical systems. When the original dynamical systems are difficult to be resolved, symbolic dynamics can provide a promising direction. By symbolic dynamical systems, we mean here the space of sequences $\Sigma_2 = \{s: s = (s_0 s_1 s_2 \dots), s_i = 0 \text{ or } s_i = 1 \text{ for all } i\}$ along with the shift map defined on it. Symbolic dynamics are trembling with maps on sets. We now introduce some basic definitions, lemma, and theorem, which are essential to this paper.

Definition 2.1:

A map $k: X \rightarrow Y$ is a conjugacy (topological) from $f_1: X \rightarrow X$ to $f_2: Y \rightarrow Y$ if

- (i) k is a homomorphism from X to Y and
- (ii) $k f_1 = f_2 k$ or equivalently $f_2 = k f_1 k^{-1}$ or $f = k^{-1} f_2 k$.

If the map k is continuous onto, they are said to be semi-conjugate (topologically). If f_2 is topologically semi-conjugated to f_1 , f_1 having a dense set of periodic points and being topologically transitive, then f_2 is a topologically transitive and has a dense set of periodic points.

Semi-conjugacy and conjugacy are more commonly related to a commuted diagram, as given in the following Figure 2.1

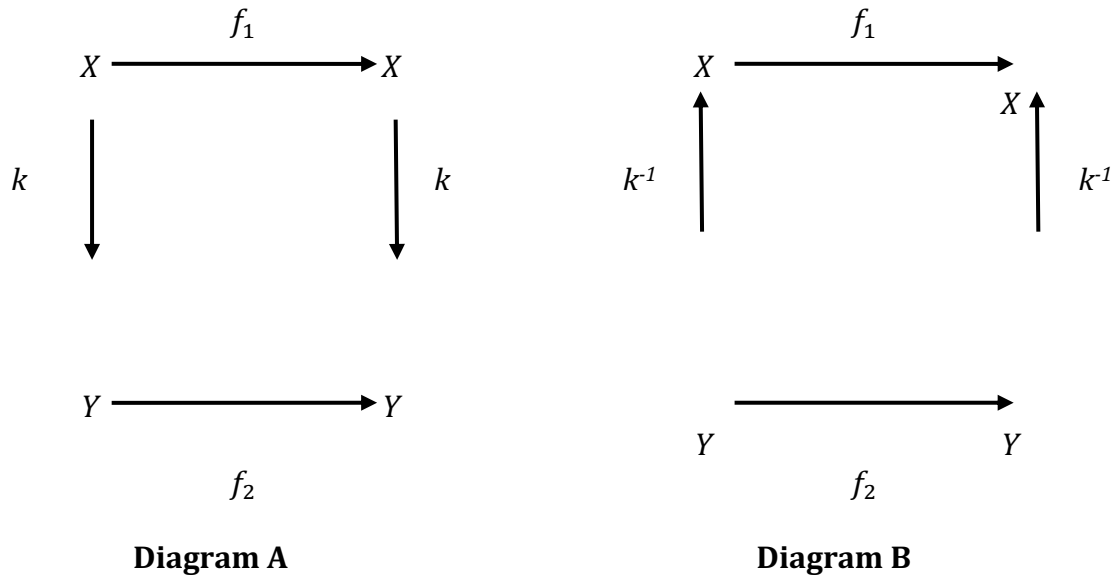


Figure 2.1: The illustration of semi-conjugacy and conjugacy between f_1 and f_2

As referred to dynamical systems, the word conjugacy means the similarity between the dynamical behavior of two mappings. Topological conjugacy between maps is a very powerful tool in the discussion of dynamical systems. Topological conjugacy is a significant concept in dynamical systems. It is an essential tool that can be used to make predictions toward the behavior of a dynamical system up to comparing it with another dynamical system whose specific properties are known [7].

Definition 2.2:

Consider a set S which is nonempty is said to be a Cantor set if the following conditions are satisfied: (i) S is bounded and closed, (ii) S contains no intervals, (iii) Every point in S is a limit point of S .

Definition 2.3:

The sequence space $\sum_n = \{(s_0 s_1 s_2 \dots)\} | s_i = \{0, 1, 2, \dots, n\}$ for all i . That is the sequence space $\sum_n$ is all the infinite sequences of $\{0, 1, 2, \dots, n\}$.

Definition 2.4:

Define a projection map $\pi: \sum_2 \rightarrow [0, 1]$ by $\pi(s_0 s_1 \dots s_n) = \sum_{n=0}^{\infty} \frac{s_n}{2^{n+1}}$

Lemma 2.1:

Consider $s, t \in \sum_2$ and suppose $s_i = t_i$ for $i = 0, 1, \dots, n$. Then, $d(s, t) = \frac{1}{2^n}$. Conversely, if $d(s, t) < \frac{1}{2^n}$ then $s_i = t_i$, for $i = 0, 1, \dots, n$.

Theorem 2.1:

The sequence space $\sum_n$ is a Cantor set that means $\sum_n$ is:

- (i) Sequentially compact,
- (ii) Perfect and
- (iii) Totally disconnected.

TOPOLOGICAL CONJUGACY AND SEMI-CONJUGACY FOR ONE DIMENSIONAL MAP

In this section, we use the concept of topological conjugacy to establish the actual dynamical systems and the resulting discrete space by symbolic dynamics which are identical [5]. We know that if $h: X \rightarrow Y$ is a homeomorphism from X to Y and S is a dense subset of X then $h(S)$ is a dense subset of Y . So any conjugacy carries a dense set of orbits to a dense set of orbits. Here we see that if the dynamics of the shift space can be disclosed, then the dynamics of the actual system are inconsiderable. This section gives an example that the logistic function is topologically conjugate to the tent map.

Lemma 3.1. If f and g are conjugates (topologically) via mapping h and if p is a fixed point of f , then $h(p)$ is a fixed point of g .

Lemma 3.2. If f and g are conjugate, the period points are carried into periodic points of the same period under conjugacy.

Proposition 3.1. (Conjugate Orbits): Suppose f is conjugate to g via the homeomorphism h . Then $\{x_1, x_2, x_3, \dots, x_n\}$ is an n -cycle for f if and only if $\{h(x_1), h(x_2), \dots, h(x_n)\}$ is an n -cycle for g . Furthermore, if the underlying metric space is R in both cases, both sets are n -cycles, and f, g, h are all differentiable with h' everywhere nonzero, then the cycles have the same (weakly) attracting or repelling behavior.

From the following examples, we see that the given two functions are conjugate.

Example 3.1. Show that the dynamical systems (R, F) and (R, G) are conjugate, where $F(x) = x^2 - 2x + 2$ and $G(x) = x^2$.

Solution: We need to find a homeomorphism $h(x)$ such that $h(F(x)) = G(h(x))$ where $h(x) = ax + b$. Now we must have $a(x^2 - 2x + 2) + b = (ax + b)^2$, so that $ax^2 - 2ax + (2a + b) = a^2x^2 + 2abx + b^2$. Solving this as an identity in x produces $a = 1, b = -1$. Thus, if $h(x) = x - 1$, it is true that $h(F(x)) = (x - 1)^2 = G(h(x))$. So these two maps are indeed conjugate.

Example 3.2. Let $a > 0$. The logistic function $f(x) = ax(1 - x)$ is conjugate to a quadratic function $Q(x) = x^2 + k$ for a suitable value of k .

Solution: We have to find a conjugacy of the form $h(x) = cx + d$.

Now $hf = Qh \Leftrightarrow c(ax(1 - x)) + d = (cx + d)^2 + k \forall x$

$$\Leftrightarrow -a = c \text{ and } a = 2d \text{ and } c + d^2 = d$$

Thus $f(x) = ax(1 - x)$ is conjugate to a map $Q(x) = x^2 + k$ for $k = \frac{a}{2} - \left(\frac{a}{2}\right)^2$, the conjugacy being $y = -ax + \frac{a}{2}$.

Now we define the itinerary map π from Λ to Σ_2 that allocate x to the itinerary s of x .

Proposition 3.2. Consider a Tent map T_b where $b > 2$, then we get the itinerary map $\pi: \Lambda \rightarrow \Sigma_2$ which is a homeomorphism.

Theorem 3.1. For the shift map, σ and the Tent map $T_b (b > 2)$ restricted to $\Lambda \subset I$, the itinerary map, π is a conjugacy (topologically).

Proof: At first, we have to show that $\pi \circ T_b = \sigma \circ \pi$. Since π is one-to-one, $\bigcap_{n \geq 0} I_{s_0 s_1 \dots s_n}$ composition of a single point. So, we consider a point x in Λ may be defined unambiguously by the sequence which is nested of intervals $\bigcap_{n \geq 0} I_{s_0 s_1 \dots s_n}$ determined by the itinerary of x . Now we see that

$$\begin{aligned} T_b(I_{s_0 s_1 \dots s_n}) &= T_b(I_{s_0} \cap T_b^{-1}(I_{s_1}) \cap \dots \cap T_b^{-n}(I_{s_n})) \\ &= T_b(I_{s_0}) \cap T_b(T_b^{-1}(I_{s_1})) \cap \dots \cap T_b(T_b^{-n}(I_{s_n})) \\ &= I \cap I_{s_1} \cap T_b^{-1}(I_{s_2}) \cap \dots \cap T_b^{-n+1}(I_{s_n}) \\ &= I_{s_1} \cap T_b^{-1}(I_{s_2}) \cap \dots \cap T_b^{-n+1}(I_{s_n}) \\ &= I_{s_1 s_2 \dots s_n} \end{aligned}$$

$$\begin{aligned} \text{Hence, } \pi T_b(x) &= \pi T_b\left(\bigcap_{n=0}^{\infty} I_{s_0 s_1 \dots s_n}\right) \\ &= \pi\left(\bigcap_{n=1}^{\infty} (I_{s_1 s_2 \dots s_n})\right) \\ &= s_1 s_2 \dots s_n \\ &= \sigma \pi(x) \end{aligned}$$

Therefore, we establish that for the shift map, π is a conjugacy (topologically).

Theorem 3.2: The shift map σ on Σ_2 is conjugate [14] to the quadratic map $q_c(x) = x^2 + c$ on Λ its invariant set, when $c < -2.368$.

The following example shows that the logistic map is topologically conjugate to the tent map.

Example 3.3. The tent map T , defined by $T(x) = \begin{cases} 2x & 0 \leq x < \frac{1}{2} \\ 2(1-x) & \frac{1}{2} \leq x < 1 \end{cases}$ is conjugate to the logistic map, $F_4 = 4x(1-x)$, via the conjugacy $k(x) = \frac{2}{\pi} \sin^{-1} \sqrt{x}$.

Solution: The function k is a homeomorphism on $I = [0,1]$. Then it follows that $k^{-1}(x) = \left(\sin \frac{\pi}{2} x\right)^2$. We need to prove that $F_4(x) = k^{-1} \circ T \circ k(x)$.

First case, $0 \leq x < \frac{1}{2}$; $F_4(x) = k^{-1} \circ T \circ k(x)$

$$\begin{aligned} &= \left(\sin \left(\frac{\pi}{2} \left(\frac{4}{\pi} \sin^{-1} \sqrt{x} \right) \right) \right)^2 \\ &= \left(\sin(2 \sin^{-1} \sqrt{x}) \right)^2 \end{aligned}$$

$$\begin{aligned}
 &= \left(2\sin(\sin^{-1} \sqrt{x})\cos(\sin^{-1} \sqrt{x}) \right)^2 \\
 &= \left(2\sqrt{x}\cos(\sin^{-1} \sqrt{x}) \right)^2 \\
 &= 4x\cos^2(\sin^{-1} \sqrt{x}) \\
 &= 4x(1 - \sin^2(\sin^{-1} \sqrt{x})) \\
 &= 4x(1 - x)
 \end{aligned}$$

The second case, $\frac{1}{2} \leq x \leq 1$;

$$\begin{aligned}
 F_4(x) &= k^{-1} o T o k(x) \\
 &= \left(\sin \left(\frac{\pi}{2} \left(-\frac{4}{\pi} \sin^{-1} \sqrt{x} + 2 \right) \right) \right)^2 \\
 &= \left(\sin(-2 \sin^{-1} \sqrt{x} + \pi) \right)^2 \\
 &= \left(2\sin \left(-\sin^{-1} \sqrt{x} + \frac{\pi}{2} \right) \cos \left(-\sin^{-1} \sqrt{x} + \frac{\pi}{2} \right) \right)^2 \\
 &= \left(2\cos(\sin^{-1} \sqrt{x}) \sin(\sin^{-1} \sqrt{x}) \right)^2 \\
 &= 4x \left(\cos(\sin^{-1} \sqrt{x}) \right)^2 \\
 &= 4x(1 - x)
 \end{aligned}$$

Hence, we get, $F_4(x) = k^{-1} o T o k(x)$ for all $x \in [0,1]$.

Therefore, all topological properties of F and T are the same, and we can concentrate on the piecewise linear tent map T .

Example 3.4. The Doubling map $D(x) = \begin{cases} 2x & \text{if } 0 < x < \frac{1}{2} \\ 2x - 1, & \text{if } \frac{1}{2} < x < 1 \end{cases}$ is conjugate to logistic Map.

Proof: Let J be the homeomorphism defined by $J(x) = \sin^2 2\pi x$ and we put $\varphi(x) = (J o D o J^{-1})(x)$.

Then we have,

$$\begin{aligned}
 \varphi(x) &= J(D(J^{-1}(x))) = J\left(D\left(\frac{1}{2\pi} \sin^{-1} \sqrt{x}\right)\right), 0 \leq \frac{\sin^{-1} \sqrt{x}}{2\pi} < 1 \text{ (i. e. } 0 \leq x < 1) \\
 &= J\left(2 \times \frac{1}{2\pi} \sin^{-1} \sqrt{x}\right), 0 \leq \frac{\sin^{-1} \sqrt{x}}{2\pi} < \frac{1}{2} \text{ (i. e. } 0 \leq x \leq \frac{1}{2})
 \end{aligned}$$

$$\begin{aligned}
&= J\left(\frac{1}{\pi} \sin^{-1} \sqrt{x}\right), 0 \leq \frac{\sin^{-1} \sqrt{x}}{2\pi} < \frac{1}{2} \text{ (i. e. } 0 \leq x < \frac{1}{2}) \\
&= \left\{ \sin\left(2\pi \times \frac{1}{\pi} \sin^{-1} \sqrt{x}\right) \right\}^2 \\
&= \{ \sin(2 \sin^{-1} \sqrt{x}) \}^2 \\
&= \{ 2 \sin(\sin^{-1} \sqrt{x}) \cdot \cos(\sin^{-1} \sqrt{x}) \}^2 \\
&= (2\sqrt{x})^2 \{ 1 - \sin^2(\sin^{-1} \sqrt{x}) \} \\
&= 4x(1-x)
\end{aligned}$$

and in the other case we have,

$$\begin{aligned}
\varphi(x) &= J\left(D(J^{-1}(x))\right) \\
&= J\left(D\left(\frac{1}{2\pi} \sin^{-1} \sqrt{x}\right)\right), 0 \leq \frac{\sin^{-1} \sqrt{x}}{2\pi} < 1 \text{ (i. e. } 0 \leq x < 1) \\
&= J\left(2 \times \frac{1}{2\pi} \sin^{-1} \sqrt{x} - 1\right), 0.5 \leq \frac{\sin^{-1} \sqrt{x}}{2\pi} < 1 \text{ (i. e. } 0 \leq x < 1) \\
&= J\left(\frac{1}{\pi} \sin^{-1} \sqrt{x} - 1\right), 0.5 \leq \frac{\sin^{-1} \sqrt{x}}{2\pi} < 1 \text{ (i. e. } 0 \leq x < \frac{1}{2}) \\
&= \left\{ \sin 2\pi \left(\frac{1}{\pi} \sin^{-1} \sqrt{x} - 1\right) \right\}^2 \\
&= \{ \sin(2 \sin^{-1} \sqrt{x} - 2\pi) \}^2 \\
&= \{ \sin(2 \sin^{-1} \sqrt{x}) \}^2 \\
&= 2 \sin(\sin^{-1} \sqrt{x}) \cdot \cos(\sin^{-1} \sqrt{x})^2 \\
&= (2\sqrt{x})^2 \{ 1 - \sin^2(\sin^{-1} \sqrt{x}) \} \\
&= 4x(1-x)
\end{aligned}$$

Hence, D is conjugate to logistic map.

From the following theorem, we prove that the itinerary map, π is continuous and surjective.

Theorem 3.2. The itinerary map, π is continuous, surjective but not a homeomorphism.

Proof: We need to show that the itinerary map π is continuous, so we let $s = s_0 s_1 \dots \in \Sigma_2$ and $\varepsilon > 0$. Pick n such that $\frac{1}{2^{n+1}} < \varepsilon$. Let $t \in \Sigma_2$ satisfy $d(s, t) < \frac{1}{2^n}$. Then $s_i = t_i$ for every $i \leq n$. Therefore

$$\begin{aligned} |\pi(s) - \pi(t)| &= \left| \sum_{k=0}^{\infty} \frac{s_k - t_k}{2^{k+1}} \right| \\ &\leq \sum_{k=0}^{\infty} \left| \frac{s_k - t_k}{2^{k+1}} \right| \\ &\leq \sum_{k=n+1}^{\infty} \left| \frac{s_k - t_k}{2^{k+1}} \right| \\ &= \frac{1}{2^n} \leq \varepsilon \end{aligned}$$

Here we see that π is surjective, so π is not a homeomorphism since it is not injective. Hence π is continuous surjective but not a homeomorphism.

Theorem 3.3. π is a topologically conjugate for the doubling map, D , and the shift map, σ .

From this section, we observe that topologically conjugacy and symbolic dynamics are an illustrious join of tools to the discussion of dynamical systems.

The following example shows that the given functions are not conjugate topologically.

Example 3.5: The two maps $f, g: [0,1] \rightarrow [0,1]$ given by

$f(x) = \begin{cases} 2x & \text{if } x < 0.5 \\ -2x + 2 & \text{if } x \geq 0.5 \end{cases}$ and $g(x) = \begin{cases} 2x & \text{if } x < 0.5 \\ 2x - 1 & \text{if } x \geq 0.5 \end{cases}$ are not topologically conjugate.

Solution: A topological conjugacy consists of a homeomorphism $h: [0,1] \rightarrow [0,1]$ which in particular must send endpoints. Moreover, it must send orbits to orbits and thus, in particular fixed points to fixed points. However, the two fixed points of g are the two endpoints, whereas the two fixed points of f are one of the endpoints and the other is in the interior; thus, h cannot be a conjugacy.

Define a function $\Psi: \Lambda \rightarrow \Sigma_2$ which we exhibit is a topological conjugacy.

Let $I_0 = [0, \frac{1}{2} - \frac{\sqrt{\mu^2 - 4\mu}}{2\mu}]$ and $I_1 = [\frac{1}{2} + \frac{\sqrt{\mu^2 - 4\mu}}{2\mu}, 1]$ such that $\Lambda_1 = \{x | F(x) \text{ is in } [0,1]\} = I_0 \cup I_1$.

Theorem 4.2. The function $\Psi: \Lambda \rightarrow \Sigma_2$ is a topological conjugacy if it is satisfied the following conditions:

- (i) Ψ is one-to-one and onto
- (ii) Ψ is continuous
- (iii) $\Psi \circ h = \sigma \circ \Psi$

SYMBOLIC DYNAMICS FOR ONE DIMENSIONAL MAP

In this section, we discuss how symbolic dynamics work for one dimensional map.

We apply the coding method to a relatively simple class of maps.

Consider the family of Tent maps $T_\mu: I \rightarrow R$ ($0 < \mu < \infty$) is defined on the unit interval $I = [0,1]$ by the expression $T_\mu(x) = \left(\mu \frac{1}{2} - \left|\frac{1}{2} - x\right|\right)$.

Proposition 4.1: If $\mu > 2$, then Λ is a cantor set (Biswas and Islam 2020), where $\Lambda = I - (\cup_{k=0}^{\infty} A_k)$.

So, by symbolic dynamics, we symbolically code every $x \in \Lambda$ with unique s in $\sum_2$. We code x by looking at the j -iteration of x under T_μ for every $j \in N$. To do this, we define a map φ_x (with respect to x) from N to $\{0,1\}$ as follows

$$\varphi_x(j) := \begin{cases} 0 & \text{if } T_\mu^j(x) \in I_0 \\ 1 & \text{if } T_\mu^j(x) \in I_1 \end{cases}$$

Then the symbolic code of x is called the itinerary of x , a sequence

$$s = \varphi_x(0)\varphi_x(1)\varphi_x(2) \dots \in \sum_2$$

When we iterate x under T_μ . The itinerary of $T_\mu(x)$ is a sequence

$$t = \varphi_x(1)\varphi_x(2)\varphi_x(3) \dots = \sigma(s)$$

So, the itinerary of $T_\mu(x)$ is the shift map.

Now, we discuss symbolic Dynamics for the Doubling map.

The Doubling map, D , for example, uses the binary expansion to code every element on the interval $[0,1)$ with a sequence in $\sum_2$. As in the partition method, the coding method is significant in the dynamical study of the system because the coding of i -iteration of x under D is the i -shift of the coding of x .

Definition 4.1: The Doubling map $F: [0,1) \rightarrow [0,1)$ is defined on the interval $[0,1)$ by the expression $x \rightarrow 2x \pmod{1}$.

Theorem 4.1: Every $x \in [0,1)$ has a binary expansion as follows $x = \sum_{n=0}^{\infty} \frac{x_n}{2^{n+1}}$ where $x_n \in \{0,1\}$ for all n .

Therefore, we may code x as the sequence $s = x_0x_1 \dots \in \sum_2$. We now prospect that the coding of the 1-iteration of x under F is the shift of the coding of x . When we iterate x under F , we have $F(x) = x_0 + \sum_{n=1}^{\infty} \frac{x_n}{2^2} \pmod{1}$. If $x_0 = 0$ then $F(x) = \sum_{n=1}^{\infty} \frac{x_n}{2^2}$ and if $x_0 = 1$ then $F(x) = \sum_{n=1}^{\infty} \frac{x_n}{2^2}$. So, the coding of $F(x)$ is $x_1x_2 \dots$ the shift of the code for x .

Theorem 4.2. Suppose $c < -2.368$. Then the $q_c(x) = x^2 + c$ is chaotic on $\Lambda = \{x \in I \mid q_c^n(x) \in I, \forall n\}$

Proof: Under the above condition on c the itinerary map $S: \Lambda \rightarrow \Sigma_2$ is a conjugacy. Then $S^{-1}: \Sigma_2 \rightarrow \Lambda$ is a homeomorphism.

(i) S^{-1} carry out periodic points for σ to periodic point for $q_c(x)$. Therefore, the density proposition guarantees that the set of periodic points for q_c is dense in Λ .

(ii) If s_1 lies on a dense orbit for σ . The sequence s_1 consists of all possible blocks of 0's and 1's of length 1, length 2, and so on. Then the density proposition says that the $S^{-1}(s_1)$ lies on a dense orbit for q_c .

(iii) Claim q_c build sensitively on initial conditions. Now $\Lambda \subset I_0 \cup I_1$. Choose $b < d(I_0, I_1)$. Now let x and $y \in \Lambda$ and $x \neq y$. Since S is a homeomorphism $\Rightarrow S(x) \neq S(y)$. Therefore, these two sequences are different at some entry, say, k^{th} .

$\therefore q_c^k(x)$ and $q_c^k(y)$ lie in different I_j .

$\therefore |q_c^k(x) - q_c^k(y)| > b$

$\Rightarrow q_c$ is transitive.

Hence q_c is chaotic on the set Λ

Here we solve a problem about the Cantor set of Logistic Functions.

Problem 4.2. (Cantor set of the logistic map):

Consider the logistic map $F_\mu = \mu x(1 - x)$ on $I = [0, 1]$, where $\mu > 4$.

Note that in this case $F_\mu\left(\frac{1}{2}\right) > 1$. Since $F_\mu(0) = 0$, it follows, by the intermediate value theorem, that $\exists \alpha_0 \in \left(0, \frac{1}{2}\right)$ such that $F_\mu(\alpha_0) = 1$. Since F_μ is monotone on $\left[0, \frac{1}{2}\right]$, the interval $I_0 = [0, \alpha_0]$ of all points the left of $\frac{1}{2}$ where $F_\mu(x) \in I$. Similarly, there exists $\alpha_1 > \frac{1}{2}$ with $F_\mu(x)$ such that $F_\mu(x) \in I$ for all $x \in I_1 = [\alpha_1, 1]$ [See Figure. 4.1(a) and 4.1(b)].

Let $A_1 = I_0 \cup I_1$. Then $A_1 = \{x \in I: F_\mu(x) \in I\}$.

Then, A_2 consists of the four closed intervals $A_2 = I_{00} \cup I_{01} \cup I_{11} \cup I_{10}$ (See Fig.4.2) where

$$I_{00} = \{x: x \in I_0 \text{ and } F_\mu(x) \in I_0\},$$

$$I_{01} = \{x: x \in I_0 \text{ and } F_\mu(x) \in I_1\},$$

$$I_{11} = \{x: x \in I_1 \text{ and } F_\mu(x) \in I_1\},$$

$$I_{10} = \{x: x \in I_1 \text{ and } F_\mu(x) \in I_0\}.$$

Continuing this process, we construct $A_n = \cup I_{s_0 s_1 \dots s_{n-1}}$, where s_i is either 0 or 1, and

$$\begin{aligned} I_{s_0 s_1 \dots s_{n-1}} &= \left\{x \in I: x \in I_{s_0}, F_\mu(x) \in I_{s_1}, \dots, F_\mu^j(x) \in I_{s_j}\right\} \\ &= \bigcap_{k=0}^j F_\mu^{-k}(I_{s_k}) \\ &= I_{s_0} \cap F_\mu^{-1}(I_{s_0 s_1 \dots s_j}) \end{aligned}$$

We first note that $A_n = \{x \in I : F_\mu^n(x) \in I\}$. Furthermore,

$$I_{s_0 s_1 \dots s_{n-1}} = I_{s_0 s_1 \dots s_{n-1}} \cap F_\mu^{-n}(I_{s_n}) \subset I_{s_0 s_1 \dots s_{n-1}}$$

Hence $A_{n+1} \subset A_n$. Define the set

$$\Lambda = \bigcap_{n=1}^{\infty} A_n$$

In the discussion that follows, Λ is a Cantor set.

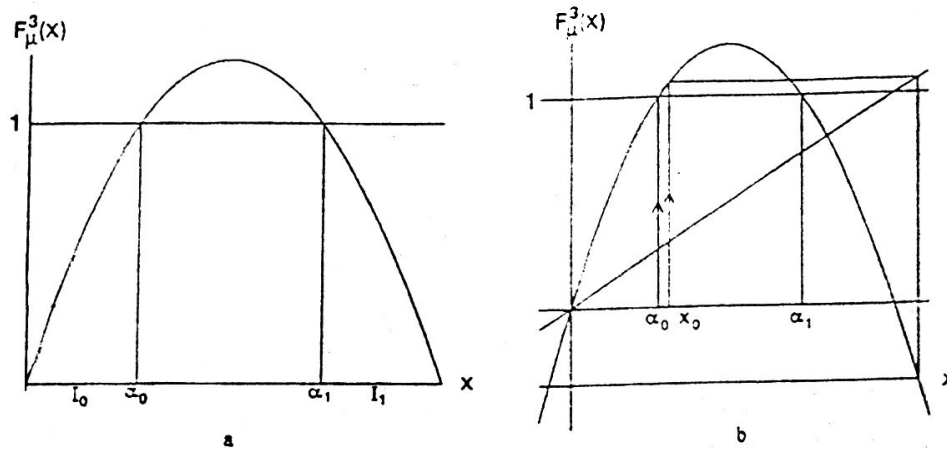


Figure 4.1: $A_1 = I_0 \cup I_1$. If $x_0 \notin I$.

We begin this task by computing points $\alpha_0 \in I_0$ and $\alpha_1 \in I_1$. Solving the equation $\mu x(1-x) = 1$, we have

$$\alpha_0 = \frac{1}{2} - \frac{\sqrt{\mu^2 - 4\mu}}{2\mu}$$

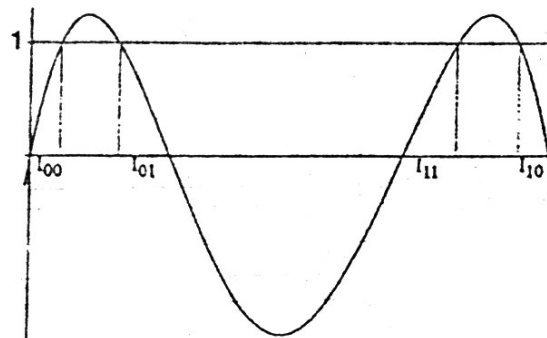


Figure 4.2: $A_2 = I_{00} \cup I_{01} \cup I_{11} \cup I_{10}$

and $\alpha_1 = \frac{1}{2} + \frac{\sqrt{\mu^2 - 4\mu}}{2\mu}$. To verify that Λ is a Cantor set, we now assume that $\mu > 2 + \sqrt{5}$. Although this is true for $\mu > 4$, the proof becomes very much involved for $4 < \mu \leq 2 + \sqrt{5}$.

CONCLUSION

In this paper, we have discussed the symbolic dynamics of one dimensional map. We have shown a conjugacy between different maps and showed a semi-conjugacy between the doubling map and the shift map. We proved the topological property of one dimensional map.

Symbolic dynamics evolved as a method to discuss standard dynamical systems; the skills and concepts have found significant applications in data storage and transmission and linear algebra. Finally, we solved a problem Cantor set of the logistic functions. We believe that several important applications of the symbolic dynamics for a two-dimensional chaotic map can be explored.

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