

More on Plastic Yield Criteria

Yehuda Partom

Retired from RAFAEL, P.O. Box 2250, Haifa, Israel

ABSTRACT

The two plastic yield criteria known as MISES and TRESCA were proposed long ago. The Mises criterion is the one usually preferred by the strength of materials community, but tests have shown that Tresca's criterion is the more accurate. We discuss here the two criteria, and based on those tests we conclude in favor of the Tresca criterion. But the two criteria are not that different, so when accuracy is not important, Mises may serve as a simple approximation to Tresca.

INTRODUCTION

In a web review R.M. Christensen [1], who is author of several strength of materials manuscripts, writes:

"All mechanics of materials books have sections on strength and failure criteria. Usually, the Mises and Tresca criteria are presented jointly with little discrimination or recommendation between them. What is more, often little else in the way of failure criteria are presented, and the non-expert can be left with the impression that these two criteria cover all or nearly all materials, and that it doesn't greatly matter which one is used".

And a little later he continues:

"The main interpretation of the Mises criterion is that it represents a critical value of the distortional energy stored in the isotropic material, while the Tresca criterion is that of critical value of the maximum shear stress in the isotropic material. Historically, the Tresca form was considered to be more fundamental of the two, but the Mises form was seen as appealing, mathematically convenient approximation to it."

Finally, after some analysis, he concludes that:

"Conclusions 2 and 3 (of the analysis) support and justify using the Mises (and not Tresca) criterion for very ductile materials, and calibrating on the critical distortional energy".

In contrast to the conclusions in [1] and to common practice, Wilkins et al. [2] obtained from tests on aluminum that the Mises criterion is not supported by those tests. The Mises criterion is usually expressed in terms of the yield (elastic-plastic) transition value of the equivalent stress. Wilkins et al. [2] performed two kinds of tests: a uniaxial stress tension test and a pure shear test. In both tests he evaluated the yield stress, and found that the two yield stresses were not equal but with values related as 1 to 0.866. According to Chocron et al. [3], there were others that observed such a difference in similar tests. Consequently, we conclude that the Mises criterion is not necessarily the better one.

In what follows we use the maximum shear stress criterion (as did Tresca) and show that it reproduces Wilkin's result. We evaluate the equivalent stress for 3 stress states: uniaxial stress, biaxial stress and pure shear. We find that for the first two the equivalent stress is higher than for the third one by the same ratio that Wilkins observed in his tests. We propose that this is because the first two states have two maximum shear planes, while the third state has only one such plane.

THREE SPECIAL STRESS STATES

The three states are: uniaxial stress, biaxial stress and pure shear. We describe those states by means of the three diagonal components of the deviator stress. These are: s_1, s_2, s_3 , where $s_1 \geq s_2 \geq s_3$. The corresponding equivalent stress is:

$$\frac{1}{3} \sigma_{eq}^2 = \frac{1}{2} (s_1^2 + s_2^2 + s_3^2) \quad (1)$$

and the maximal shear stress is: $2\tau_{max} = s_1 - s_3$.

The values of s_1, s_2, s_3 that we use for the three stress states are:

$$\begin{aligned} s_1 &= \frac{2}{\sqrt{3}} & s_2 &= -\frac{1}{\sqrt{3}} & s_3 &= -\frac{1}{\sqrt{3}} & \text{uniaxial stress} \\ s_1 &= \frac{1}{\sqrt{3}} & s_2 &= \frac{1}{\sqrt{3}} & s_3 &= -\frac{2}{\sqrt{3}} & \text{biaxial stress} \\ s_1 &= \frac{\sqrt{3}}{2} & s_2 &= 0 & s_3 &= -\frac{\sqrt{3}}{2} & \text{pure shear} \end{aligned} \quad (2)$$

and the maximal shear stresses are:

$$\begin{aligned} 2\tau_{max} &= \frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}} = \sqrt{3} & \text{uniaxial stress} \\ 2\tau_{max} &= \frac{1}{\sqrt{3}} + \frac{2}{\sqrt{3}} = \sqrt{3} & \text{biaxial stress} \\ 2\tau_{max} &= \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} = \sqrt{3} & \text{pure shear} \end{aligned} \quad (3)$$

and we see that the three maximal shear stresses are indeed equal. The equivalent stresses of the three stress states are:

$$\begin{aligned} \frac{1}{3} \sigma_{eq}^2 &= \frac{1}{2} \left(\frac{4}{3} + \frac{1}{3} + \frac{1}{3} \right) = 1 & \text{uniaxial stress} \\ \frac{1}{3} \sigma_{eq}^2 &= \frac{1}{2} \left(\frac{1}{3} + \frac{1}{3} + \frac{4}{3} \right) = 1 & \text{biaxial stress} \\ \frac{1}{3} \sigma_{eq}^2 &= \frac{1}{2} \left(\frac{3}{4} + \frac{3}{4} \right) = \frac{3}{4} & \text{pure shear} \end{aligned} \quad (4)$$

and we see that:

$$\frac{\sigma_{eq} \text{ pure shear}}{\sigma_{eq} \text{ uniaxial or biaxial stress}} = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2} = 0.866 \quad (5)$$

which is close to the test results of Wilkins et al. [2].

INTERMEDIATE STRESS STATES

We define intermediate states, between the three discrete states of the previous section, by:

$$\beta = \frac{s_2}{s_1} \quad -\frac{1}{2} < \beta < 1 \quad (6)$$

and the range of change of β can be seen in Fig. 1.

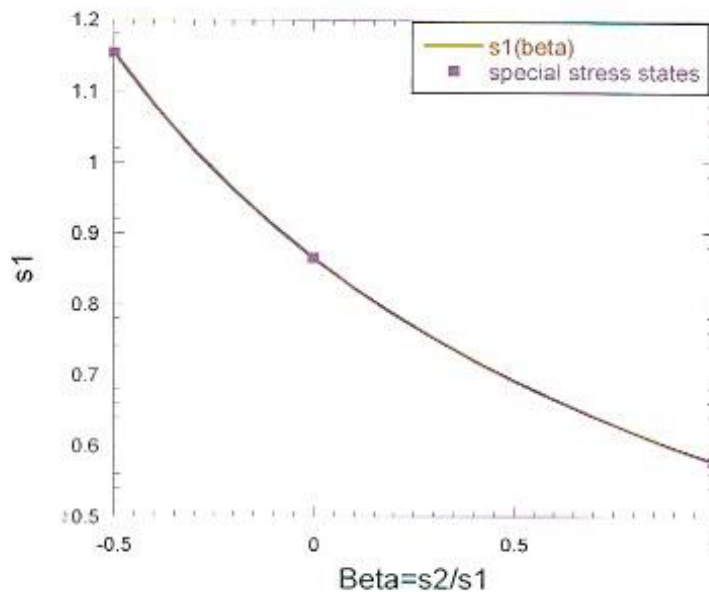


Figure 1. Variation of the large deviatoric stress component s_1 as function of the parameter $\beta = s_2/s_1$. The blue squares are the special stress states of the previous section

The stress components of the above stress deviator as function of the parameter β are: s_1 ; $s_2 = \beta s_1$; $s_3 = -(1 + \beta)s_1$, and the maximal shear stress τ_{max} is:

$$2\tau_{max} = s_1 + (1 + \beta)s_1 = (2 + \beta)s_1 \quad (7)$$

For the maximal shear stress to be the same as in Eq. (3) we need that:

$$(2 + \beta)s_1 = \sqrt{3} \quad \therefore s_1 = \frac{\sqrt{3}}{2 + \beta} \quad (8)$$

The equivalent stress is given as function of β by:

$$\frac{1}{3}\sigma_{eq}^2 = \frac{3}{(2 + \beta)^2} (1 + \beta + \beta^2) \quad (9)$$

and we show the relation $\sigma_{eq}(\beta)$ in Fig. 2.

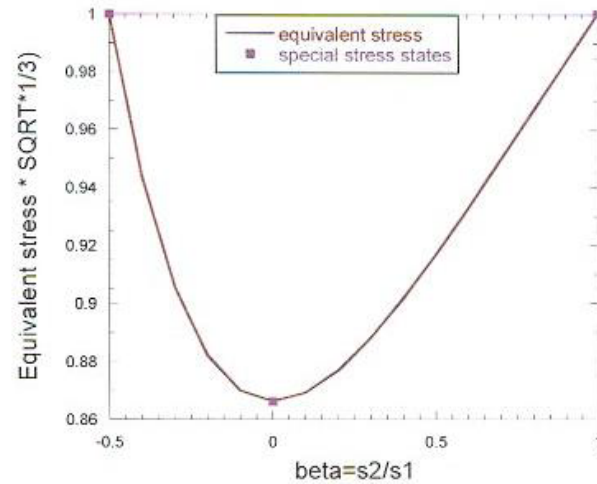


Figure 2. Change of the equivalent stress as function of $\beta=s_1/s_2$. The blue squares are the special stress states of the previous section

We see from Fig. 2 that $\sigma_{eq}(\beta)$ is a non-symmetric parabola with the pure shear state at the minimum point.

A possible explanation to the results given in Eq. (4) and to the experimental findings of Wilkins et al. [2] is that the uniaxial and biaxial stress states have two planes of maximum shear stress, while the pure shear state has only one such plane.

COMPARISON TO TRESCA'S CRITERION

As stated in the introduction, Tresca is a maximum shear stress criterion. Also, when plotting the Tresca equivalent stress as function of the Lode angle, one gets a straight line for each of the 60° sections of the Lode angle. Next, we show that the Tresca criterion plot of the equivalent stress as function of the parameter β is identical to the curve of Fig. 2.

According to Y. Bai and T. Wierzbicki [4], the Lode angle θ is given as function of β by:

$$\cos(3\theta) = \frac{27}{2} \frac{\beta(1+\beta)}{3^{1.5}(1+\beta+\beta^2)^{1.5}} \quad (10)$$

In Fig. 3 we show the relation $\theta(\beta)$ according to Eq. (10).

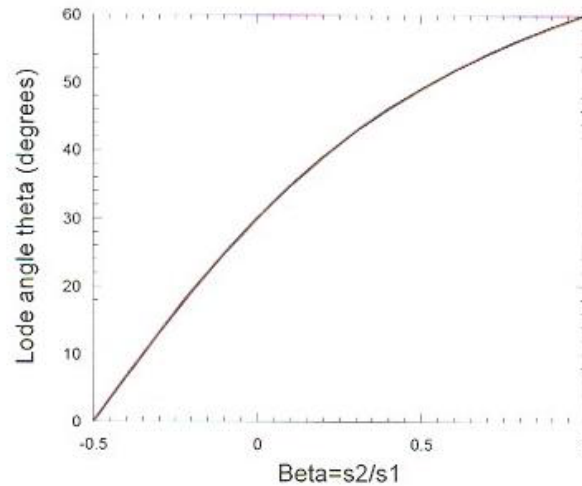


Figure 3. Lode angle (θ) as function of the parameter β

Using Eq. (10) we can compute the equivalent stress on the Tresca criterion surface. As discussed in the introduction, the Tresca surface is an equilateral hexagon. It has a straight line between $\theta=0$ and $\theta=60^\circ$, and so on until $\theta=360^\circ$. Between $\theta=0$ and $\theta=60^\circ$ we therefore get:

$$\sigma_{eq} = \sigma_{eq}^0 \frac{\cos 30^\circ}{\cos(30^\circ - \theta)} = \sigma_{eq}^0 \frac{\sqrt{3}/2}{\cos(30^\circ - \theta)} \quad (11)$$

where σ_{eq}^0 is the equivalent stress at $\theta=0$.

In Fig. 4 we compare σ_{eq} from Eq. (11) (Tresca) to what we got in Fig. 2.

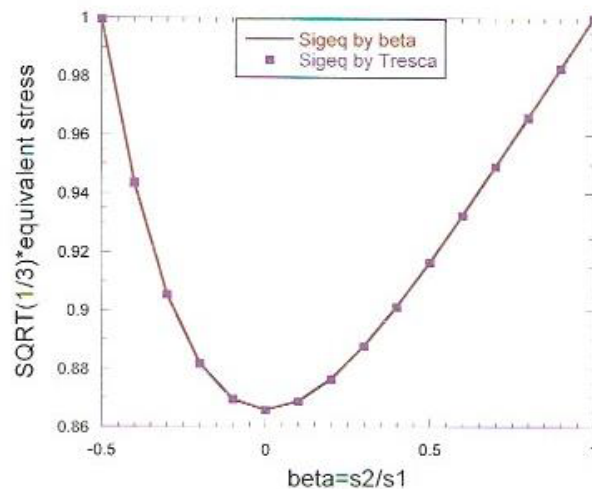


Figure 4. The equivalent stress as function of the parameter $\beta=s_2/s_1$. The curve of Fig. 2 (line) compared to the Tresca criterion (points)

We see from Fig. 4 that the equivalent stress from the two computations is the same. This is so because in both computations the shear stress is maximal. It makes a lot of sense to maximize

the shear stress as a condition to the onset of plastic flow, because shear stress is what drives dislocation motion through the material crystal, and dislocation motion creates plastic flow.

The question that still remains is, why does the strength of materials community prefer to use the Mises criterion upon the Tresca criterion? Possible answers are: 1) the Mises criterion is simpler to use with computations; 2) the corners in the Tresca surface may cause difficulties with some computations; 3) the difference between the two surfaces is not that big.

SUMMARY

We discuss here the criterion for plastic yield in solids. Two such criteria have been proposed long ago, the Mises and the Tresca criteria. The Mises criterion represents plastic yielding in terms of a constant equivalent stress surface, while the Tresca criterion is based on maximizing the shear stress. During the years the Mises criterion became the criterion of choice with the mechanics of materials community. But tests have shown that, in contrast to the Mises criterion, the equivalent stress is not constant upon yielding. For instance, the yielding equivalent stress for pure shear is 0.866 of the yielding equivalent stress for uniaxial stress.

Accordingly, we assume here (as did Tresca) that a material starts to flow plastically when the shear stress is maximal. We then evaluate the equivalent stress upon yielding for different shear situations (Fig. 2). It turns out that for pure shear the equivalent stress upon yielding is the smallest, and for uniaxial and biaxial stress states it is the largest. Also, the ratio between the smallest and the largest equivalent stress values is 0.866, as obtained in tests. To explain this behavior, we propose that it comes about because pure shear has a single maximum shear plane, while uniaxial and biaxial stress states have two such planes. It follows that contrary to common practice, Tresca is more appropriate than Mises. But the difference is rather small, and using Mises is usually more convenient.

References

1. R.M. Christensen, how do Mises and Tresca fit in, Web review, FailurCriteria.com, (2019).
2. M.L. Wilkins et al., Cumulative strain damage model of ductile fracture: simulation and prediction of engineering fracture tests, UCRL-53058. LLNL report (1980).
3. S. Chocron et al., A new plasticity and failure model for ballistic application, Int. J. of Impact Eng. 38, 755-764 (2011).
4. Y. Bai and T. Wierzbicki, Int J. Plasticity 24, 1071-1096 (2008).