

Unitary Theory of Quantum Mechanics and General Relativity

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ABSTRACT

Essentially, in this paper we propose a new description of the quantum dynamics by two relativistic propagation wave packets, in the two conjugated spaces, of the coordinates and of the momentum. Compared to the Schrödinger-Dirac equation, which describes a free particle by a wave function continuously expanding in time, considered as the amplitude of a probabilistic distribution of this particle, the new equations describe a free particle as an invariant distribution of matter propagating in the two spaces, as it should be. Matter quantization arises from the equality of the integral of the matter density with the mass describing the dynamics of this density in the phases of the wave packets. In this description, the classical Lagrange and Hamilton equations are obtained as the group velocities of the two wave packets in the coordinate and momentum spaces. When to the relativistic Lagrangian we add terms with a vector potential conjugated to coordinates, as in the Aharonov-Bohm effect, and a scalar potential conjugated to time, we obtain the Lorentz force and the Maxwell equations as characteristics of the quantum dynamics. In this framework, the conventional Schrödinger-Dirac equations of a quantum particle in an electromagnetic field obtain additional terms explicitly depending on velocity, as is expected in the framework of relativistic theory. Such a particle wave function takes the form of a rapidly varying wave, with the frequency corresponding to the rest energy, modulated by the electric potential $U(x^i)$ where this particle is confined, the vector potential $A^j(x^i, t)$ of the electromagnetic radiation, and the gravitational potential. In the gravitational field of a black hole, the velocity of a quantum particle is directed to the Schwarzschild boundary from the outside and from the inside, without passing this boundary. When the very strong gravitational field of a large agglomeration of matter has spherical symmetry with a radius smaller than the Schwarzschild radius, the quantum particles in the central region are crushed up to quasi-uniform distributions of matter, exploding with velocities much larger than the light velocity, towards the Schwarzschild boundary, but decelerating in the gravitational field and reaching this boundary only in an infinite time. The matter of a black hole has the tendency to concentrate around the Schwarzschild boundary, with a radius proportional to its mass. In this framework, we believe that our universe is a huge black hole, with the explosion of its central matter as the Big Bang-inflation process, and an increasing radius on account of an absorption of matter from the Total Universe, where our universe has been formed. In a gravitational wave, the time-space interval in the first-order approximation is the product of a gravitational invariant including the amplitude tensor, and a coordinate invariant including the polarization vector. The gravitational invariant

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describes a rotation of the metric tensor around the direction of the propagation of this wave, with spin 2, as a gravitational quantum called graviton. The coordinate invariant describes a matter rotation with the spins $\frac{1}{2}$ for Fermions, and 1 for Bosons. From the new dynamic equations, for a free particle in the coordinate and momentum spaces, we reobtain the two basic equations of the quantum field theory, but with a change of sign, and an additional term depending on momentum, to the rest mass as the eigenvalue of these equations. However, when these eigenvalues are eliminated, the wave function takes the form of a wave packet of spinors of the same form as in the conventional quantum field theory, with a normalization volume as the integral of the ratio of the energy to the rest energy, over the momentum domain which gives finite dimensions to the quantum particle, as a finite distribution of matter in the coordinate space.

INTRODUCTION

In the conventional quantum mechanics, a quantum particle with an inertial mass M is described by the Schrödinger equation¹

$$i\hbar \frac{\partial}{\partial t} \psi(\vec{r}, t) = H(\vec{r}, \vec{p}) \psi(\vec{r}, t), \quad (1)$$

which means that the Hamiltonian

$$H(\vec{r}, \vec{p}) = T(\vec{p}) + U(\vec{r}) = \frac{\vec{p}^2}{2M} + U(\vec{r}) \quad (2)$$

is an operator of the form

$$H(\vec{r}, \vec{p}) = i\hbar \frac{\partial}{\partial t}. \quad (3)$$

With this operator and the momentum operator

$$\vec{p} = -i\hbar \frac{\partial}{\partial \vec{r}}, \quad (4)$$

one obtains the explicit form of the Schrödinger equation,

$$i\hbar \frac{\partial}{\partial t} \psi(\vec{r}, t) = \left[-\frac{\hbar^2}{2M} \frac{\partial^2}{\partial \vec{r}^2} + U(\vec{r}) \right] \psi(\vec{r}, t). \quad (5)$$

With the energy E as a Hamiltonian eigenvalue, this equation takes the form of the steady state Schrödinger equation

$$H(\vec{p}, \vec{r}) \psi(\vec{r}) = \left[-\frac{\hbar^2}{2M} \frac{\partial^2}{\partial \vec{r}^2} + U(\vec{r}) \right] \psi(\vec{r}) = E \psi(\vec{r}), \quad (6)$$

which is in very good agreement with the experimental data in numerous applications of atomic, nuclear, and condensed matter physics. With a solution

$$|\psi(\vec{r}, t)\rangle = e^{-\frac{i}{\hbar} H(\vec{r}, \vec{p})t} |\psi_0(\vec{r})\rangle, \quad (7)$$

of the time dependent Schrödinger equation (1), for a time dependent operator in the Heisenberg picture,

$$A(t) = e^{\frac{i}{\hbar}H(\vec{r},\vec{p})t} A e^{-\frac{i}{\hbar}H(\vec{r},\vec{p})t}, \quad (8)$$

we obtain the Heisenberg dynamic equation

$$\frac{\partial}{\partial t} A(t) = \frac{i}{\hbar} [H(\vec{r},\vec{p}), A(t)]. \quad (9)$$

This equation for the coordinate and momentum in a direction x , seems to be in agreement with the Hamilton equations,

$$\begin{aligned} \frac{\partial}{\partial t} \langle x(t) \rangle &= \frac{\langle p_x(t) \rangle}{M} \\ \frac{\partial}{\partial t} \langle p_x(t) \rangle &= - \left\langle \frac{\partial U(x,t)}{\partial x} \right\rangle. \end{aligned} \quad (10)$$

However, in fact this is not true, since for a null potential $U(x,t)=0$ the solution of the Schrödinger equation (1) is a plane wave, while the mean value of the coordinate $\langle x(t) \rangle$ does not make any sense – the Hamilton equations also correctly describe the free states. According to the momentum operator (4) and the Hamiltonian operator (2), we might consider the solution of the Schrödinger equation (1) as a propagation state, of the form of a wave packet of the momentum eigenstates,

$$\psi(\vec{r},t) = \frac{1}{(2\pi\hbar)^{3/2}} \int \varphi(\vec{p},t) e^{\frac{i}{\hbar}[\vec{p}\vec{r}-H(\vec{p},\vec{r})t]} d^3\vec{p} = \frac{1}{(2\pi\hbar)^{3/2}} e^{-\frac{i}{\hbar}U(\vec{r})t} \int \varphi(\vec{p},t) e^{\frac{i}{\hbar}[\vec{p}\vec{r}-T(\vec{p})t]} d^3\vec{p}, \quad (11)$$

which, in fact, stands at the basis of the derivation of this Schrödinger equation¹. With the inverse Fourier transform

$$\varphi(\vec{p},t) = \frac{1}{(2\pi\hbar)^{3/2}} \int \psi(\vec{r},t) e^{-\frac{i}{\hbar}[\vec{p}\vec{r}-H(\vec{p},\vec{r})t]} d^3\vec{r} = \frac{1}{(2\pi\hbar)^{3/2}} e^{\frac{i}{\hbar}T(\vec{p})t} \int \psi(\vec{r},t) e^{-\frac{i}{\hbar}[\vec{p}\vec{r}-U(\vec{r})t]} d^3\vec{r}, \quad (12)$$

the dynamics of a quantum particle take the form of two wave packets propagating in the two conjugated spaces of the coordinates and momentum. We notice that the wave packet (11) of a realistic quantum particle, with small dimensions in the coordinate and momentum spaces, with the group velocity

$$\frac{d\vec{r}}{dt} = \frac{\partial T(\vec{p})}{\partial \vec{p}} = \frac{\partial}{\partial \vec{p}} H(\vec{p},\vec{r}), \quad (13)$$

is in agreement with one of the Hamilton equations, as the wave packet (12) with the group velocity

$$\frac{d\vec{p}}{dt} = \frac{\partial U(\vec{r})}{\partial \vec{r}} = \frac{\partial}{\partial \vec{r}} H(\vec{p}, \vec{r}) \quad (14)$$

is in contradiction with the other Hamilton equation – a minus is missing.

QUANTUM DYNAMIC EQUATION FOR PROPAGATION STATES

For dynamics in full agreement with the Hamilton equations, instead of the wave packets (11) and (12) obtained from the Schrödinger equation (1) with momentum (4), we consider the wave packets

$$\begin{aligned} \psi(\vec{r}, t) &= \frac{1}{(2\pi\hbar)^{3/2}} \int \varphi(\vec{p}, t) e^{\frac{i}{\hbar} \{ \vec{p}\vec{r} - [T(\vec{p}) - U(\vec{r})]t \}} d^3\vec{p} \\ &= \frac{1}{(2\pi\hbar)^{3/2}} \int \varphi(\vec{p}, t) e^{\frac{i}{\hbar} [\vec{p}\vec{r} - L(\vec{p}, \vec{r})t]} d^3\vec{p} \\ \varphi(\vec{p}, t) &= \frac{1}{(2\pi\hbar)^{3/2}} \int \psi(\vec{r}, t) e^{-\frac{i}{\hbar} \{ \vec{p}\vec{r} - [T(\vec{p}) - U(\vec{r})]t \}} d^3\vec{r} \\ &= \frac{1}{(2\pi\hbar)^{3/2}} \int \psi(\vec{r}, t) e^{-\frac{i}{\hbar} [\vec{p}\vec{r} - L(\vec{p}, \vec{r})t]} d^3\vec{r}, \end{aligned} \quad (15)$$

with the group velocities for small particles

$$\begin{aligned} \frac{d\vec{r}}{dt} &= \frac{\partial T(\vec{p})}{\partial \vec{p}} = \frac{\partial}{\partial \vec{p}} H(\vec{p}, \vec{r}) \\ \frac{d\vec{p}}{dt} &= -\frac{\partial U(\vec{r})}{\partial \vec{r}} = -\frac{\partial}{\partial \vec{r}} H(\vec{p}, \vec{r}). \end{aligned} \quad (16)$$

With the Lagrangian as a function of the Hamiltonian,

$$\begin{aligned} L(\vec{p}, \vec{r}) &= \vec{p}\dot{\vec{r}} - H(\vec{p}, \vec{r}) = \frac{\vec{p}^2}{M} - H(\vec{p}, \vec{r}) = \frac{\vec{p}^2}{2M} - U(\vec{r}) \\ &= T(\vec{p}) - U(\vec{r}), \end{aligned} \quad (17)$$

the wave functions (15) take a form

$$\begin{aligned}
 \psi(\vec{r}, t) &= \frac{1}{(2\pi\hbar)^{3/2}} \int \varphi(\vec{p}, t) e^{\frac{i}{\hbar} \{ \vec{p}\vec{r} - [\vec{p}\dot{\vec{r}} - H(\vec{p}, \vec{r})] t \}} d^3 \vec{p} \equiv e^{\frac{i}{\hbar} \vec{p}\vec{r}} \psi_t(\vec{r}, t) \\
 &= \frac{1}{(2\pi\hbar)^{3/2}} \int \varphi(\vec{p}, t) e^{\frac{i}{\hbar} [\vec{p}\vec{r} - (\vec{p}\dot{\vec{r}} - E)t]} d^3 \vec{p} \\
 \varphi(\vec{p}, t) &= \frac{1}{(2\pi\hbar)^{3/2}} \int \psi(\vec{r}, t) e^{-\frac{i}{\hbar} \{ \vec{p}\vec{r} - [\vec{p}\dot{\vec{r}} - H(\vec{p}, \vec{r})] t \}} d^3 \vec{r} \equiv e^{-\frac{i}{\hbar} \vec{p}\vec{r}} \varphi_t(\vec{p}, t) \\
 &= \frac{1}{(2\pi\hbar)^{3/2}} \int \psi(\vec{r}, t) e^{-\frac{i}{\hbar} [\vec{p}\vec{r} - (\vec{p}\dot{\vec{r}} - E)t]} d^3 \vec{r},
 \end{aligned} \tag{18}$$

where we distinguish the time dependent factor $\psi_t(\vec{r}, t)$ of the wave function, satisfying the dynamic equation

$$i\hbar \frac{\partial}{\partial t} \psi_t(\vec{r}, t) = [\vec{p}\dot{\vec{r}} - H(\vec{p}, \vec{r})] \psi_t(\vec{r}, t), \tag{19}$$

which, with the Hamiltonian (2) and the momentum (4),

$$i\hbar \left(\frac{\partial}{\partial t} + \dot{\vec{r}} \frac{\partial}{\partial \vec{r}} \right) \psi_t(\vec{r}, t) = - \left[-\frac{\hbar^2}{2M} \frac{\partial^2}{\partial \vec{r}^2} + U(\vec{r}) \right] \psi_t(\vec{r}, t), \tag{20}$$

takes a form somehow similar to the Schrödinger equation (1) for the time dependent factor of a propagation wave function,

$$i\hbar \frac{d}{dt} \psi_t(\vec{r}, t) = -H(\vec{p}, \vec{r}) \psi_t(\vec{r}, t). \tag{21}$$

We notice that with a time dependent linearization of the coordinates $\vec{r} = \dot{\vec{r}}t$, valid for a free particle, the propagation state (18) is reduced to its time dependent factor, generated by an evolution operator

$$U_t = e^{\frac{i}{\hbar} Ht}. \tag{22}$$

It is interesting that by this linearization, the wave function (18) is reduced to the solution of a Schrödinger equation of the form (1), with a nonsignificant change of sign, only because we preferred to keep the minus sign for the momentum operator (4), as the Schrödinger equation, symmetric with the momentum, does not depend on this sign. However, this linearization means a null potential, which means that, in fact, the dynamic equation obtained in this way is not the Schrödinger equation (1), but only a particular form, with the null potential energy.

QUANTUM PARTICLE AS A DISTRIBUTION OF MATTER

The wave equations (15) suggest a relativistic form, with the relativistic Lagrangian²

$$L(x^\alpha, v^\alpha) = -Mc^2 \sqrt{g_{\alpha\beta} v^\alpha v^\beta}, \tag{23}$$

where $g_{\alpha\beta}$ is the metric tensor depending on the gravitational potential, and v^α, v^β are the time-space velocities of the Fourier components,

$$v^\alpha = \frac{dx^\alpha}{ds} = \frac{dx^\alpha}{cd\tau}, \quad (24)$$

in the proper time τ as a parameter representing the travelled time-space interval

$$ds \equiv cd\tau = \sqrt{g_{\alpha\beta} dx^\alpha dx^\beta}. \quad (25)$$

We obtain the wave functions

$$\begin{aligned} \psi(x^i, \tau) &= \frac{1}{(2\pi\hbar)^{3/2}} \int \varphi(P^j, \tau) e^{\frac{i}{\hbar}[P^j x^j - L(x^\alpha, v^\alpha)\tau]} d^3P \\ \varphi(P^j, \tau) &= \frac{1}{(2\pi\hbar)^{3/2}} \int \psi(x^i, \tau) e^{-\frac{i}{\hbar}[P^j x^j - L(x^\alpha, v^\alpha)\tau]} d^3x, \end{aligned} \quad (26)$$

with the proper time τ as a parameter of motion in the local system of curvilinear coordinates

$$x = (x^\alpha) = (x^0 = ct, x^i). \quad (27)$$

We notice that the differential of the time dependent phases of these wave functions,

$$\begin{aligned} L(x^\alpha, v^\alpha) d\tau &= -Mc^2 d\tau \sqrt{g_{\alpha\beta} v^\alpha v^\beta} = -Mc \sqrt{g_{\alpha\beta} dx^\alpha dx^\beta} = -Mc ds = -Mc^2 d\tau \\ &= -Mc^2 \sqrt{g_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta} dt \end{aligned} \quad (28)$$

is proportional to the differential time-space interval, which means that the relativistic principle of the invariance of the time-space interval is equivalent to a quantum relativistic principle of the invariance of the time dependent phase, which determines the particle dynamics.

By dividing expression (25) by ds , we obtain the fundamental equation for velocities:

$$g_{\alpha\beta} v^\alpha v^\beta = 1. \quad (29)$$

With this equation, the symmetry of the metric tensor, and the canonical momentum

$$\begin{aligned} P^j &= \frac{\partial L}{c \partial v^j} = -Mc^2 \frac{\partial}{c \partial v^j} \sqrt{g_{\alpha\beta} v^\alpha v^\beta} = -Mc \frac{1}{2\sqrt{g_{\alpha\beta} v^\alpha v^\beta}} \frac{\partial}{\partial v^j} (g_{j\beta} v^j v^\beta + g_{\alpha j} v^\alpha v^j) \\ &= -Mc \frac{g_{j\beta} v^\beta + g_{\alpha j} v^\alpha}{2\sqrt{g_{\alpha\beta} v^\alpha v^\beta}} = -Mc g_{j\mu} v^\mu, \end{aligned} \quad (30)$$

the group velocity of the wave packet (26) for a small particle in the coordinate space is

$$\begin{aligned}
\frac{d}{d\tau} x^j &= \frac{\partial L}{\partial P^j} = \frac{\partial \left(-Mc^2 \sqrt{g_{\alpha\beta} v^\alpha v^\beta} \right)}{\partial \left(-Mc g_{j\mu} v^\mu \right)} \\
&= \frac{c}{2\sqrt{g_{\alpha\beta} v^\alpha v^\beta}} \frac{\partial}{\partial (g_{j\mu} v^\mu)} \left(g_{j\beta} v^j v^\beta + g_{\alpha j} v^\alpha v^j \right) = c \frac{2v^j}{2\sqrt{g_{\alpha\beta} v^\alpha v^\beta}} = cv^j,
\end{aligned} \tag{31}$$

as the group velocity in the momentum space is of the form of the Lagrange equation,

$$\frac{d}{d\tau} P^j = \frac{\partial}{\partial x^j} L(x^\alpha, v^\alpha) \tag{32}$$

With a metric tensor as a function of the curvilinear coordinates describing the gravitational field, the time derivative of the momentum takes the form of the gradient of the scalar potential of this field,

$$\frac{d}{d\tau} P^j = \frac{-Mc^2 v^\alpha v^\beta}{2\sqrt{g_{\alpha\beta} v^\alpha v^\beta}} \frac{\partial}{\partial x^j} g_{\alpha\beta} = -\frac{1}{2} Mc^2 v^\alpha v^\beta g_{\alpha\beta,j} \tag{33}$$

From equation (31) for a very narrow momentum spectrum δP^j , we obtain the propagation

velocity cv_M^j of a Fourier component of the wave function (26) as a velocity $\frac{d}{d\tau} x_M^j$ of its maximum value. We notice that with the constant Lagrangian (28), as the time dependent phase of a Fourier component^{3,4}, the wave function (26) describes a distribution not depending on time, $\psi(x^j, \tau) = \psi(x^j - x_M^j, 0)$, but only on the propagation factor with the phase $P^j x^j$, and the momentum spectrum, which, as an integral of such a distribution, is also time independent $\varphi(P^j, \tau) = \varphi(P^j)$. These integrations will be performed in the last section, in the momentum one wave approximation of the quantum field theory. This suggests the interpretation of this wave function as an invariant distribution of matter, with a sufficiently narrow momentum spectrum propagating with the group velocity (31). From the fundamental equation (29), we notice that the invariant time dependent phases of the wave functions (26) with the Lagrangian (23), do not depend on momentum, but only on the proper time τ , which means that, at any time, the width Δx^j of the particle is given only by the product $\Delta x^j \cdot \Delta P^j$ of the coordinate dependent phase. These wave functions represent constant distributions of matter propagating in time, unlike solution (11) of Schrödinger equation (1), with the time dependent phase proportional to the Hamiltonian, which leads to a rapid increase in the time of the width of the

particle wave function, $\Delta \vec{r} = \sqrt{\langle \vec{r}^2 \rangle - \langle \vec{r} \rangle^2}$, as a function of the momentum spectrum width $\Delta \vec{p} = \sqrt{\langle \vec{p}^2 \rangle - \langle \vec{p} \rangle^2}$, $\Delta \vec{r}(t) = 3 \frac{\Delta \vec{p}}{M} t$.

On the other hand, since the covariant derivative of the metric tensor is null,

$$g_{\mu\nu;\sigma} = 0, \quad (34)$$

from the fundamental equation (29) for any matter element we obtain the relation

$$v^\alpha_{;\sigma} v_\alpha = 0, \quad (35)$$

which, by multiplication with v^σ ,

$$v^\alpha_{;\sigma} v^\sigma v_\alpha = 0, \quad (36)$$

means that this equation describes a covariant acceleration $v^\alpha_{;\sigma} v^\sigma$ of a matter element perpendicular to the velocity v_α . More precisely, with an acceleration A^μ , induced by an external (nongravitational) field to any matter element, we obtain a change in the geodesic trajectory depending on the parameter τ^2 ,

$$v^\mu_{;\nu} v^\nu = -\Gamma^\mu_{\nu\sigma} v^\nu v^\sigma + A^\mu. \quad (37)$$

With this expression we find that an external (nongravitational) field induces a covariant acceleration,

$$A^\mu = (v^\mu_{;\nu} + \Gamma^\mu_{\nu\sigma} v^\sigma) v^\nu = v^\mu_{;\nu} v^\nu, \quad (38)$$

which, with (36), is perpendicular to the particle velocity in its proper time,

$$A^\mu v_\mu = 0. \quad (39)$$

From the group velocity (31) with this equation, we find that the acceleration induced by the relativistic Lagrangian (28) under the action of an external field on a matter element is perpendicular to its velocity. In this way, the wave function (26) with the Lagrangian (28) is a packet of rapidly varying waves, with a large frequency component proportional to the rest energy, any external field changing the perpendicular direction of the propagation velocity on an external field only as a weak modulation by small additional terms to this Lagrangian.

For such a distribution of matter we consider a positively defined density

$$\rho(x^i, \tau) = M_0 |\psi(x^i, \tau)|^2, \quad (40)$$

with an amplitude function normalized to 1,

$$\int |\psi(x^i, \tau)|^2 d^3x = 1. \quad (41)$$

For this amplitude function, we consider a Fourier series expansion describing the matter dynamics according to general theory of relativity, of the form (26) with a time dependent phase of the form (28). Such dynamics can be conceived only as a function with inertial characteristics M of the Lagrangian, as the wave functions (26) are density amplitude functions of a matter distribution with a total mass M_0 . We consider the equality of these two differently defined masses,

$$\int \rho(x^i, \tau) d^3x = M_0 \int |\psi(x^i, \tau)|^2 d^3x = M_0 = M, \quad (42)$$

as a quantization rule of the matter distribution. In this framework, a quantum particle^{5,6} can be understood as an element of a continuous distribution matter, with the mass and dynamics given by the dynamic characteristics of the matter density, according to the general theory of relativity.

QUANTUM MATTER CONSERVATION

In the previous section, we showed that a quantum particle is a distribution of matter propagating in space with an unaltered shape in a free motion, with a mass determined by density dynamics characteristics. We consider a quantum particle of mass M , with the density scalar field ρ in two systems of reference,

$$\int \rho(x^{\mu'}) dx^{0'} dx^{1'} dx^{2'} dx^{3'} = \int \rho(x^\mu) J dx^0 dx^1 dx^2 dx^3 = M, \quad (43)$$

where

$$J = \text{Det}(x_{,\alpha}^{\mu'}) \quad (44)$$

is the Jacobian. We notice that the elements of this Jacobian are elements of a tensor transformation, which for the metric tensor is

$$g_{\alpha\beta} = x_{,\alpha}^{\mu'} x_{,\beta}^{\nu'} g_{\mu'\nu'}. \quad (45)$$

From this relation, we obtain the Jacobian as a function of the determinants of the metric tensor in the two systems of reference,

$$J = \frac{\sqrt{-g}}{\sqrt{-g'}}. \quad (46)$$

With this expression, from equation (43) we obtain the transformation relation of the matter density, as a scalar field with the scalar density² $\sqrt{-g}$,

$$\int \rho(x^{\mu'}) \sqrt{-g'} dx^{0'} dx^{1'} dx^{2'} dx^{3'} = \int \rho(x^\mu) \sqrt{-g} dx^0 dx^1 dx^2 dx^3. \quad (47)$$

For a density of the matter of a quantum particle, we consider the matter flow four-vector

$$J^\mu = \rho v^\mu, \quad (48)$$

which satisfies the conservation relation of a null covariant divergence,

$$J^\mu_{;\mu} = 0. \quad (49)$$

By using the expression of the covariant derivative as a function of the ordinary derivative, we obtain the covariant divergence

$$J^\mu_{;\mu} = J^\mu_{,\mu} + \Gamma^\mu_{\nu\mu} J^\nu = J^\nu_{,\nu} + \Gamma^\mu_{\nu\mu} J^\nu. \quad (50)$$

With an index interchange of the symmetric metric tensor in the Christoffel symbol,

$$\begin{aligned}\Gamma_{\nu\mu}^{\mu} &= \frac{1}{2} g^{\mu\lambda} \left(\underline{g_{\lambda\nu,\mu}} + g_{\lambda\mu,\nu} - \underline{g_{\mu\nu,\lambda}} \right) = \frac{1}{2} g^{\mu\lambda} g_{\lambda\mu,\nu} \\ &= \frac{1}{2} g^{-1} g_{,\nu} = \frac{1}{2} (-g)^{-1} (-g)_{,\nu} = \frac{(\sqrt{-g})_{,\nu}}{\sqrt{-g}},\end{aligned}\quad (51)$$

the condition (49) takes the form of a null ordinary divergence,

$$J^{\mu}_{;\mu} \sqrt{-g} = (J^{\nu} \sqrt{-g})_{,\nu} = 0 \quad (52)$$

With the Gauss formula, by integrating this equation over a volume V with the scalar density $\sqrt{-g}$, we obtain the conservation relation, of the time variation of the matter contained in a volume V by a matter flow through the surface Σ_V of this volume,

$$\left(\int_V \rho v^0 \sqrt{-g} d^3x \right)_{,0} = - \oint_{\Sigma_V} \rho v^m \sqrt{-g} d^2x^m, \quad (53)$$

which, in the nonrelativistic case, $v^m \ll v^0 \simeq 1$,

$$J^0 = \rho v^0 \simeq \rho, \quad (J^m) = \rho (v^m) \doteq \rho \frac{\vec{v}}{c}, \quad (54)$$

takes the classical form of the matter conservation,

$$\frac{\partial}{\partial t} \int_V \rho d^3\vec{r} = - \oint_{\Sigma_V} \rho \vec{v} d^2\vec{r} \quad (55)$$

From the conservation equation (53), for the matter of a volume $V(\tau) = \{x^{j'}(\tau), x^{j_1}(\tau) < x^{j'} < x^j(\tau)\}$ of a quantum particle propagating as a constant distribution of matter in the interval $\{x^{j_1}(\tau), x^{j_2}(\tau)\}$, we obtain a null matter velocity through the surface $\{x^j(\tau)\}$. In the conservation equation (53), the velocities v^m of a differential matter element with the coordinates x^m , for a constant distribution of matter with a density (40), is the resultant of the velocities v^α of the Fourier components for these coordinates, which is equal to the propagation velocity of the wave packet (26).

QUANTUM PARTICLE IN ELECTROMAGNETIC FIELD

The wave functions (26) describe a black particle, a black matter, observable only by gravitational effects. We now consider a quantum particle in a local system (27), with the wave functions (26) which, with the Lagrangian (28), are of the form

$$\begin{aligned}\psi(x^i, t) &= \frac{1}{(2\pi\hbar)^{3/2}} \int \varphi(P^j, t) e^{\frac{i}{\hbar} [P^j x^j - L(x^\alpha, \dot{x}^\alpha) t]} d^3 P \\ \varphi(P^j, t) &= \frac{1}{(2\pi\hbar)^{3/2}} \int \psi(x^i, t) e^{-\frac{i}{\hbar} [P^j x^j - L(x^\alpha, \dot{x}^\alpha) t]} d^3 x.\end{aligned}\quad (56)$$

The time dependent phase of the wave function (56) of a particle, as the amplitude function of the density (40), takes a linear time dependent form with a constant Lagrangian function (28)

$$\int_0^t L\left(x^\alpha(t'), \frac{dx^\alpha(t')}{dt'}\right) dt' = L\left(x^\alpha, \frac{dx^\alpha}{dt}\right) t$$

by the linearization of the invariant integral⁰, in the spatial domain of this particle. Such a linearization of the time dependent phase is justified for a realistic quantum particle that is usually much smaller than the curvature nonuniformities of the physical space, as no coordinate variation of the metric elements in this integral is significant, and the velocities are constant. When the quantum particle has a charge, coupling this particle with a field, the differential time dependent phase (28) obtains additional terms proportional to this charge. For an electric charge e interacting with an electromagnetic field, the time dependent phase (28) obtains additional terms with a potential conjugated to the coordinates, which we call vector potential $\vec{A}(\vec{r}, t)$, and a scalar potential conjugated to time,

which we call electric potential $U(\vec{r})$,

$$\begin{aligned}L(\vec{r}, \dot{\vec{r}}, t) dt &= -Mc^2 \sqrt{1 - \frac{\dot{\vec{r}}^2}{c^2}} dt + e\vec{A}(\vec{r}, t) d\vec{r} - eU(\vec{r}) dt \\ &= \left(-Mc^2 \sqrt{1 - \frac{\dot{\vec{r}}^2}{c^2}} + e\vec{A}(\vec{r}, t) \dot{\vec{r}} - eU(\vec{r})\right) dt,\end{aligned}\quad (57)$$

where the first electromagnetic term describes the Aharonov-Bohm effect⁷. Here we considered a time independent electric potential and a time dependent vector potential as in the atom-field interaction^{8,9,10,11,12,13,14}. For the Lagrangian we obtain the canonical momentum

$$\vec{P} = \frac{\partial}{\partial \dot{\vec{r}}} L(\vec{r}, \dot{\vec{r}}, t) = \frac{M\dot{\vec{r}}}{\sqrt{1 - \frac{\dot{\vec{r}}^2}{c^2}}} + e\vec{A}(\vec{r}, t) = \vec{p} + e\vec{A}(\vec{r}, t), \quad (58)$$

as a sum of the mechanical momentum $\vec{p}$ with the electromagnetic momentum $e\vec{A}(\vec{r}, t)$. With this momentum, we obtain the Hamiltonian

$$H(\vec{P}, \vec{r}, t) = \vec{P} \dot{\vec{r}} - L(\vec{r}, \dot{\vec{r}}, t) = \frac{Mc^2}{\sqrt{1 - \frac{\dot{\vec{r}}^2}{c^2}}} + eU(\vec{r}) = E(\vec{r}, \dot{\vec{r}}), \quad (59)$$

with the energy as a conservative function. With the identity

$$\frac{M^2 c^2}{1 - \frac{\dot{\vec{r}}^2}{c^2}} = \vec{p}^2 + M^2 c^2, \quad (60)$$

from (59) we obtain the canonical form of the Hamiltonian,

$$H(\vec{P}, \vec{r}, t) = c\sqrt{M^2 c^2 + \vec{p}^2} + eU(\vec{r}) = c\sqrt{M^2 c^2 + [\vec{P} - e\vec{A}(\vec{r}, t)]^2} + eU(\vec{r}) \quad (61)$$

For a quantum particle much smaller than the significant variation dimensions of the electromagnetic potentials, from (58) we obtain the electromagnetic force acting on this particle,

$$\vec{F}_e = \frac{d}{dt} \vec{p} = \frac{d}{dt} \vec{P} - e \frac{d}{dt} \vec{A}(\vec{r}, t) = \frac{d}{dt} \vec{P} - e \frac{\partial}{\partial t} \vec{A}(\vec{r}, t) - e \left(\dot{\vec{r}} \frac{\partial}{\partial \vec{r}} \right) \vec{A}(\vec{r}, t) \quad (62)$$

In this expression, we consider the time variation of the canonical momentum as the velocity of a wave packet in the momentum space of the form (56), which is of the form of the Lagrange equation. With (57), this variation is

$$\frac{d}{dt} \vec{P} = \frac{\partial}{\partial \vec{r}} L(\vec{r}, \dot{\vec{r}}, t) = e \frac{\partial}{\partial \vec{r}} [\vec{A}(\vec{r}, t) \dot{\vec{r}}] - e \frac{\partial}{\partial \vec{r}} U(\vec{r}) \quad (63)$$

With the first term of (63) from the vector formula

$$\dot{\vec{r}} \times \left[\frac{\partial}{\partial \vec{r}} \times \vec{A}(\vec{r}, t) \right] = \frac{\partial}{\partial \vec{r}} [\dot{\vec{r}} \vec{A}(\vec{r}, t)] - \left(\dot{\vec{r}} \frac{\partial}{\partial \vec{r}} \right) \vec{A}(\vec{r}, t) \quad (64)$$

and the expressions

$$\begin{aligned} \vec{E}(\vec{r}, t) &= -\frac{\partial}{\partial t} U(\vec{r}) - \frac{\partial}{\partial t} \vec{A}(\vec{r}, t) \\ \vec{B}(\vec{r}, t) &= \frac{\partial}{\partial \vec{r}} \times \vec{A}(\vec{r}, t), \end{aligned} \quad (65)$$

of the electric and magnetic fields, from (62) we obtain the Lorentz force

$$\vec{F}_e = e\vec{E}(\vec{r}, t) + e\dot{\vec{r}} \times \vec{B}(\vec{r}, t) \quad (66)$$

From (65), we obtain the Faraday-Maxwell law of the electromagnetic induction,

$$\frac{\partial}{\partial \vec{r}} \times \vec{E}(\vec{r}, t) = -\frac{\partial}{\partial t} \vec{B}(\vec{r}, t) \quad (67)$$

and the Gauss-Maxwell law of the magnetic field,

$$\frac{\partial}{\partial \vec{r}} \vec{B}(\vec{r}, t) = 0 \quad (68)$$

With the gauge condition,

$$\frac{\partial}{\partial \vec{r}} \vec{A}(\vec{r}, t) = 0 \quad (69)$$

we obtain the Gauss-Maxwell law of the electric field, with a source $\rho(\vec{r}, t)$,

$$\frac{\partial}{\partial \vec{r}} \vec{E}(\vec{r}, t) = -\frac{\partial^2}{\partial \vec{r}^2} U(\vec{r}, t) = \frac{\rho(\vec{r}, t)}{\varepsilon_0}, \quad (70)$$

where ε_0 is a dimensional constant called electric permittivity. For field propagation we consider regions with uniform charge densities. For a sphere S with surface Σ_s , we obtain a flux equation

$$\oint_{\Sigma_s} \vec{E}(\vec{r}, t) d\vec{\sigma} = \frac{1}{\varepsilon_0} \int_S \rho(\vec{r}, t) dV \quad (71)$$

with the electric field source $\rho(\vec{r}, t)$ as the density of a charge q , according to the Coulomb formula

$$\vec{E}(\vec{r}, t) = \frac{q}{4\pi\varepsilon_0 r^2} \frac{\vec{r}}{r} = \frac{1}{4\pi\varepsilon_0 r^2} \frac{\vec{r}}{r} \int_S \rho(\vec{r}, t) dV. \quad (72)$$

For a uniform distribution of charge, from (70) we obtain

$$\frac{\partial}{\partial \vec{r}} \left[\frac{\partial}{\partial \vec{r}} \vec{E}(\vec{r}, t) \right] = \frac{1}{\varepsilon_0} \frac{\partial}{\partial \vec{r}} \rho = 0, \quad (73)$$

which leads to the vector formula

$$\frac{\partial}{\partial \vec{r}} \times \left[\frac{\partial}{\partial \vec{r}} \times \vec{E}(\vec{r}, t) \right] = \frac{\partial}{\partial \vec{r}} \left[\frac{\partial}{\partial \vec{r}} \vec{E}(\vec{r}, t) \right] - \frac{\partial^2}{\partial \vec{r}^2} \vec{E}(\vec{r}, t) = -\frac{\partial^2}{\partial \vec{r}^2} \vec{E}(\vec{r}, t). \quad (74)$$

With this expression, from the Faraday-Maxwell law (67), we obtain the Laplacian of the electric field as a function of the magnetic field

$$\frac{\partial^2}{\partial \vec{r}^2} \vec{E}(\vec{r}, t) = \frac{\partial}{\partial t} \frac{\partial}{\partial \vec{r}} \times \vec{B}(\vec{r}, t). \quad (75)$$

For the electric field we consider an equation of propagation that, for the physical consistency of the theory, has a velocity equal to the maximum velocity c of a particle according to the theory of relativity, and, for a region with a uniform charge density ρ , a dissipation rate γ ,

$$\frac{\partial^2}{\partial \vec{r}^2} \vec{E}(\vec{r}, t) = \frac{1}{c^2} \left[\frac{\partial^2}{\partial t^2} \vec{E}(\vec{r}, t) + \gamma \frac{\partial}{\partial t} \vec{E}(\vec{r}, t) \right]. \quad (76)$$

From (75) and (76), by a time integration with a dimensional constant μ_0 called magnetic permeability, as a function of the electric permittivity and the light velocity,

$$c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}, \quad (77)$$

we obtain the Ampère-Maxwell law of the magnetic circuit of the form

$$\frac{1}{\mu_0} \frac{\partial}{\partial \vec{r}} \times \vec{B}(\vec{r}, t) = \varepsilon_0 \frac{\partial}{\partial t} \vec{E}(\vec{r}, t) + \varepsilon_0 \gamma \vec{E}(\vec{r}, t) \quad (78)$$

For a more explicit understanding of the dissipative term of this equation, we consider the last term of this equation as the total time derivative of the electric field,

$$\frac{d}{dt} \vec{E}(\vec{r}, t) \equiv \frac{\partial}{\partial t} \vec{E}(\vec{r}, t) + \left(\dot{\vec{r}} \frac{\partial}{\partial \vec{r}} \right) \vec{E}(\vec{r}, t) = \frac{\partial}{\partial t} \vec{E}(\vec{r}, t) + \gamma \vec{E}(\vec{r}, t), \quad (79)$$

where $\dot{\vec{r}}$ is the charge velocity field induced by the electric field $\vec{E}$ in its direction,

$$\dot{\vec{r}} \times \vec{E}(\vec{r}, t) = 0 \quad (80)$$

With the curl of this null vector and the Gauss-Maxwell law of the electric field (70),

$$\frac{\partial}{\partial \vec{r}} \times \left[\dot{\vec{r}} \times \vec{E}(\vec{r}, t) \right] = \dot{\vec{r}} \left[\frac{\partial}{\partial \vec{r}} \vec{E}(\vec{r}, t) \right] - \left(\dot{\vec{r}} \frac{\partial}{\partial \vec{r}} \right) \vec{E}(\vec{r}, t) = \dot{\vec{r}} \frac{\rho}{\varepsilon_0} - \left(\dot{\vec{r}} \frac{\partial}{\partial \vec{r}} \right) \vec{E}(\vec{r}, t) = 0, \quad (81)$$

from equation (79) we obtain

$$\dot{\vec{r}} \frac{\rho}{\varepsilon_0} = \gamma \vec{E}(\vec{r}, t) \quad (82)$$

We obtain a charge velocity field proportional to the electric field, with a charge mobility μ coefficient,

$$\dot{\vec{r}} = \mu \vec{E}(\vec{r}, t), \quad (83)$$

as the dissipation rate of the electric field is proportional to this mobility and the charge density,

$$\gamma = \frac{\mu}{\varepsilon_0} \rho \quad (84)$$

With this expression and (83), for a field current,

$$\vec{j}(\vec{r}, t) = \rho \dot{\vec{r}} = \mu \rho \vec{E}(\vec{r}, t) = \varepsilon_0 \gamma \vec{E}(\vec{r}, t) = \sigma \vec{E}(\vec{r}, t), \quad (85)$$

the Ampère-Maxwell law of the magnetic circuit (78) takes the conventional form

$$\frac{1}{\mu_0} \frac{\partial}{\partial \vec{r}} \times \vec{B}(\vec{r}, t) = \vec{j}(\vec{r}, t) + \varepsilon_0 \frac{\partial}{\partial t} \vec{E}(\vec{r}, t) \quad (86)$$

From this equation with the Gauss-Maxwell law for the electric field (70), we obtain the equation of the charge conservation,

$$\frac{\partial}{\partial \vec{r}} \cdot \vec{j}(\vec{r}, t) = - \frac{\partial}{\partial t} \rho(\vec{r}, t) \quad (87)$$

In this way, we obtained the dynamics of the electromagnetic field from the condition of interaction with a quantum particle, for a velocity of propagation of this field equal to the maximum velocity of the particle according to the theory of relativity. The universally accepted Maxwell equations are in agreement with the expression (56) of the wave functions of a quantum particle, as distributions of matter with equal velocities of its differential elements. The conservation of the matter with mass of a quantum particle, and of the electric charge, show that, although we can detect only the total mass M and charge e of a particle, the mass and the charge densities are characteristics of the continuous matter distribution of the particle.

QUANTUM PARTICLE IN ELECTROMAGNETIC AND GRAVITATIONAL FIELDS

The dynamics of a quantum particle in an electromagnetic field is described by equation (21) with Hamiltonian (61), and we consider Dirac's form¹⁵

$$H(\vec{p}, \vec{r}) = c(\alpha_0 Mc + \alpha_1 p_1 + \alpha_2 p_2 + \alpha_3 p_3) + eU(\vec{r}), \quad (88)$$

with Dirac's operators

$$\alpha_0 = \begin{pmatrix} \hat{1} & 0 \\ 0 & -\hat{1} \end{pmatrix}, \alpha_1 = \begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix}, \alpha_2 = \begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix}, \alpha_3 = \begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix}, \quad (89)$$

as functions of the Pauli spin operators,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (90)$$

which satisfy the commutation relations

$$\{\alpha_i, \alpha_j\} = 2\delta_{ij}, \quad \{\sigma_i, \sigma_j\} = 2\delta_{ij}, \quad \sigma_i \sigma_j = i\delta_{ijk} \sigma_k. \quad (91)$$

With the mechanical momentum as a function of the canonical momentum operator (58),

$$\vec{p} = -i\hbar \frac{\partial}{\partial \vec{r}} - e\vec{A}(\vec{r}, t), \quad (92)$$

for the wave function four-vector,

$$\psi_i = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \quad \psi_1 = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}, \quad \psi_2 = \begin{pmatrix} \varphi_3 \\ \varphi_4 \end{pmatrix}, \quad (93)$$

we obtain a system of four coupled dynamic equations,

$$\begin{aligned} -i\hbar \left(\frac{\partial}{\partial t} + \dot{x} \frac{\partial}{\partial x} + \dot{y} \frac{\partial}{\partial y} + \dot{z} \frac{\partial}{\partial z} \right) \varphi_1(\vec{r}, t) &= (Mc^2 + eU(\vec{r})) \varphi_1(\vec{r}, t) \\ + c \left(-i\hbar \frac{\partial}{\partial x} - eA_x(\vec{r}, t) \right) \varphi_4(\vec{r}, t) &- ic \left(-i\hbar \frac{\partial}{\partial y} - eA_y(\vec{r}, t) \right) \varphi_4(\vec{r}, t) + c \left(-i\hbar \frac{\partial}{\partial z} - eA_z(\vec{r}, t) \right) \varphi_3(\vec{r}, t) \end{aligned} \quad (94a)$$

$$\begin{aligned}
 & -i\hbar \left(\frac{\partial}{\partial t} + \dot{x} \frac{\partial}{\partial x} + \dot{y} \frac{\partial}{\partial y} + \dot{z} \frac{\partial}{\partial z} \right) \varphi_2(\vec{r}, t) = (Mc^2 + eU(\vec{r})) \varphi_2(\vec{r}, t) \\
 & + c \left(-i\hbar \frac{\partial}{\partial x} - eA_x(\vec{r}, t) \right) \varphi_3(\vec{r}, t) + ic \left(-i\hbar \frac{\partial}{\partial y} - eA_y(\vec{r}, t) \right) \varphi_3(\vec{r}, t) - c \left(-i\hbar \frac{\partial}{\partial z} - eA_z(\vec{r}, t) \right) \varphi_4(\vec{r}, t)
 \end{aligned} \quad (94b)$$

$$\begin{aligned}
 & -i\hbar \left(\frac{\partial}{\partial t} + \dot{x} \frac{\partial}{\partial x} + \dot{y} \frac{\partial}{\partial y} + \dot{z} \frac{\partial}{\partial z} \right) \varphi_3(\vec{r}, t) = (-Mc^2 + eU(\vec{r})) \varphi_3(\vec{r}, t) \\
 & + c \left(-i\hbar \frac{\partial}{\partial x} - eA_x(\vec{r}, t) \right) \varphi_2(\vec{r}, t) - ic \left(-i\hbar \frac{\partial}{\partial y} - eA_y(\vec{r}, t) \right) \varphi_2(\vec{r}, t) + c \left(-i\hbar \frac{\partial}{\partial z} - eA_z(\vec{r}, t) \right) \varphi_1(\vec{r}, t)
 \end{aligned} \quad (94c)$$

$$\begin{aligned}
 & -i\hbar \left(\frac{\partial}{\partial t} + \dot{x} \frac{\partial}{\partial x} + \dot{y} \frac{\partial}{\partial y} + \dot{z} \frac{\partial}{\partial z} \right) \varphi_4(\vec{r}, t) = (-Mc^2 + eU(\vec{r})) \varphi_4(\vec{r}, t) \\
 & + c \left(-i\hbar \frac{\partial}{\partial x} - eA_x(\vec{r}, t) \right) \varphi_1(\vec{r}, t) + ic \left(-i\hbar \frac{\partial}{\partial y} - eA_y(\vec{r}, t) \right) \varphi_1(\vec{r}, t) - c \left(-i\hbar \frac{\partial}{\partial z} - eA_z(\vec{r}, t) \right) \varphi_2(\vec{r}, t),
 \end{aligned} \quad (94d)$$

which, compared to other similar relativistic quantum dynamic equations^{5,6}, include an explicit dependence on the velocity $(\dot{x}, \dot{y}, \dot{z})$.

To take into account a gravitational wave, we consider the Lagrangian (57) in a system of curvilinear coordinates with a time-space diagonalization,

$$L(x^\alpha, \dot{x}^\alpha) = -Mc^2 \sqrt{g_{00} + g_{ij} \dot{x}^i \dot{x}^j} - eU(x^i) + eA^j(x^\mu) \dot{x}^j. \quad (95)$$

With the momentum

$$P^j = \frac{\partial L}{\partial \dot{x}^j} = \frac{-Mcg_{ij}\dot{x}^i}{\sqrt{g_{00} + g_{ik}\dot{x}^i\dot{x}^k}} + eA^j(x^\mu), \quad (96)$$

we obtain the Hamiltonian

$$H = cP^j \dot{x}^j - L(x^\alpha, \dot{x}^\alpha) = \frac{Mc^2 g_{00}}{\sqrt{g_{00} + g_{ij} \dot{x}^i \dot{x}^j}} + eU(x^i). \quad (97)$$

With the expression

$$\begin{aligned}
 \frac{M^2 c^2 g_{00}^2}{g_{00} + g_{ij} \dot{x}^i \dot{x}^j} &= g_{00} \left(\frac{-M^2 c^2 g_{ij} \dot{x}^i \dot{x}^j}{g_{00} + g_{ij} \dot{x}^i \dot{x}^j} + M^2 c^2 \right) = g_{00} \left(\frac{Mc\dot{x}^j (P^j - eA^j)}{\sqrt{g_{00} + g_{ij} \dot{x}^i \dot{x}^j}} + M^2 c^2 \right) \\
 &= g_{00} \left(\delta_i^j \frac{Mc\dot{x}^i (P^j - eA^j)}{\sqrt{g_{00} + g_{ij} \dot{x}^i \dot{x}^j}} + M^2 c^2 \right) = g_{00} \left(g^{jk} g_{ki} \frac{Mc\dot{x}^i (P^j - eA^j)}{\sqrt{g_{00} + g_{ij} \dot{x}^i \dot{x}^j}} + M^2 c^2 \right) \\
 &= g_{00} \left[M^2 c^2 - g^{jk} (P^j - eA^j) (P^k - eA^k) \right],
 \end{aligned} \quad (98)$$

from the Hamiltonian (97), we obtain the canonical form

$$H = c\sqrt{g_{00}\left(M^2c^2 - g^{ij}\left[P^i - ecA^i(x^\mu)\right]\left[P^j - ecA^j(x^\mu)\right]\right)} + eU(x^i), \quad (99)$$

depending on the metric elements g_{00} and g^{ij} , as functions of the gravitational potential. In the nonrelativistic case, of a velocity much smaller than the light velocity, as the fundamental relation (29) is

$$\sqrt{g_{00} + g_{ij}\dot{x}^i\dot{x}^j} \simeq 1, \quad (100)$$

the momentum (96) takes the simpler form

$$P^j \simeq -Mcg_{ij}\dot{x}^i + eA^j(x^\mu). \quad (101)$$

With this expression, we obtain the Lagrangian (95) as a function of the coordinates and momentum,

$$\begin{aligned} L &= -Mc^2\sqrt{g_{00} + g_{ij}\dot{x}^i\dot{x}^j} - eU(x^i) + ecA^j(x^\mu)\delta_k^j\dot{x}^k \\ &= -Mc^2\sqrt{g_{00} + g_{ij}\dot{x}^i\dot{x}^j} - e\left[U(x^i) + g^{jk}A^j(x^\mu)\frac{P^k - eA^k(x^\mu)}{M}\right], \end{aligned} \quad (102)$$

as the wave functions (56) are of the explicit form

$$\begin{aligned} \psi(x^i, t) &= \frac{1}{(2\pi\hbar)^{3/2}} \int \varphi(P^j, t) e^{\frac{i}{\hbar}\left\{P^j x^j + e\left[U(x^i) + g^{jk}A^j(x^\mu)\frac{P^k - eA^k(x^\mu)}{M}\right]t\right\}} e^{\frac{i}{\hbar}Mc^2\sqrt{g_{00} + g_{ij}\dot{x}^i\dot{x}^j}t} d^3P \\ \varphi(P^j, t) &= \frac{1}{(2\pi\hbar)^{3/2}} \int \psi(x^i, t) e^{-\frac{i}{\hbar}\left\{P^j x^j + e\left[U(x^i) + g^{jk}A^j(x^\mu)\frac{P^k - eA^k(x^\mu)}{M}\right]t\right\}} e^{-\frac{i}{\hbar}Mc^2\sqrt{g_{00} + g_{ij}\dot{x}^i\dot{x}^j}t} d^3x, \end{aligned} \quad (103)$$

of rapidly oscillating waves with a very large frequency corresponding to the rest mass, slightly modulated by the electromagnetic field with the modulation frequency

$$\omega_E = \frac{1}{\hbar}\left[eU(x^i) + eg^{jk}A^j(x^\mu)\frac{P^k - eA^k(x^\mu)}{M}\right] \simeq \frac{1}{\hbar}eU(x^i). \quad (104)$$

With the group velocity for a particle much smaller than the dimensions of the significant field variations,

$$\frac{dx^i}{dt} = c\dot{x}^i = -\hbar\frac{\partial\omega_E}{\partial P^i} = -\frac{e}{M}g^{ik}A^k(x^\mu), \quad (105)$$

from (101) we obtain the wave vector of a particle in electromagnetic field,

$$k^j_E = \frac{1}{\hbar} P^j = \frac{1}{\hbar} \left[-Mc g_{ij} \dot{x}^i + e A^j(x^\mu) \right] = 2 \frac{e}{\hbar} A^j(x^\mu) \quad (106)$$

In (103), the interaction of the quantum particle with the gravitational wave is described by the gravitational potential g_{00} , as a solution of the gravitational wave equation

$$g^{\mu\nu} g_{\rho\sigma, \mu\nu} = 0, \quad (107)$$

where the electromagnetic momentum compared to the rest momentum is neglected, $eA^k \ll Mc$,

$$Mc^2 \sqrt{g_{00} + g_{ij} \dot{x}^i \dot{x}^j} = Mc^2 \sqrt{g_{00} + \frac{e^2}{M^2 c^2} A^k(x^\mu) A^l(x^\mu)} \approx Mc^2 \sqrt{g_{00}} \quad (108)$$

We notice that the wave functions (103) describe the wave functions of a quantum particle as a rapidly varying wave, with a frequency $\frac{1}{\hbar} Mc^2$, modulated with a frequency $\frac{e}{\hbar} U(x^\mu)$ of the electric potential, where this particle is confined, a momentum $eA^k(x^\mu)$ of the electromagnetic field acting on the particle, and a frequency of the gravitational wave described by the metric tensor g_{00} as a gravitational potential.

DYNAMICS OF A QUANTUM PARTICLE IN A GRAVITATIONAL FIELD AND BLACK HOLES

We consider a quantum particle in a referential with a time-space diagonalization, as the Lagrangian (23) is of the form

$$L(x^j, v^j) = -Mc^2 \sqrt{g_{00} + g_{ij} v^i v^j}, \quad (109)$$

with the momentum (30),

$$P^j = -Mc g_{ij} v^i \quad (110)$$

We obtain a Lagrangian

$$\begin{aligned} L(x^j, v^j) &= -Mc^2 \sqrt{g_{00} + g_{ij} v^i v^j} = -Mc^2 \sqrt{g_{00} - \frac{P^j v^j}{Mc}} \\ &= -Mc^2 \sqrt{g_{00} - \frac{\delta_i^j P^j v^i}{Mc}} = -Mc^2 \sqrt{g_{00} - \frac{g_{ik} g^{kj} P^j v^i}{Mc}} \\ &= -Mc^2 \sqrt{g_{00} + \frac{g^{kj} P^j P^k}{M^2 c^2}}, \end{aligned} \quad (111)$$

for an explicit form of propagation wave functions (26), as wave packets of the momentum eigenstates,

$$\begin{aligned}\psi(x^i, \tau) &= \frac{1}{(2\pi\hbar)^{3/2}} \int \varphi(P^j, \tau) e^{\frac{i}{\hbar} [P^j x^j + c \sqrt{g_{00} M^2 c^2 + g^{ij} P^i P^j} \tau]} d^3 P \\ \varphi(P^j, \tau) &= \frac{1}{(2\pi\hbar)^{3/2}} \int \psi(x^i, \tau) e^{-\frac{i}{\hbar} [P^j x^j + c \sqrt{g_{00} M^2 c^2 + g^{ij} P^i P^j} \tau]} d^3 x.\end{aligned}\quad (112)$$

In these expressions of the dynamics of the matter of a quantum particle the proper time τ is a parameter, as a good approximation of the local referential time $t \approx \tau$ only for a velocity much smaller than the light velocity,

$$\dot{x}^j = \frac{dx^j}{dx^0} = \frac{dx^j}{cdt} \ll \dot{x}^0 = 1 \quad (113)$$

For an arbitrary velocity, we consider the dynamics of the whole particle, according to the geodesic equation

$$\frac{d^2 x^i}{dt^2} = -c^2 \Gamma_{\alpha\beta}^i \dot{x}^\alpha \dot{x}^\beta, \quad (114)$$

which, with the Christoffel symbols as functions of the elements of the metric tensor, is

$$\frac{d^2 x^i}{dt^2} = -\frac{1}{2} c^2 g^{i\lambda} (g_{\lambda\alpha, \beta} + g_{\lambda\beta, \alpha} - g_{\alpha\beta, \lambda}) \dot{x}^\alpha \dot{x}^\beta. \quad (115)$$

In approximation (113) of a velocity much smaller than the light velocity with a time-space diagonalization, this equation takes the form of the gradient of the metric tensor element g_{00} ,

$$\frac{d^2 x^i}{dt^2} = \frac{1}{2} c^2 g^{ii} g_{00, i}. \quad (116)$$

In a central field, described by the Schwarzschild metric elements,

$$\begin{aligned}g_{00} &= 1 - \frac{2m}{r}, & g_{11} &= -\left(1 - \frac{2m}{r}\right)^{-1}, & g_{22} &= -r^2, & g_{33} &= -r^2 \sin^2 \theta \\ g^{00} &= \left(1 - \frac{2m}{r}\right)^{-1}, & g^{11} &= -\left(1 - \frac{2m}{r}\right), & g^{22} &= -r^{-2}, & g^{33} &= -r^{-2} \sin^{-2} \theta,\end{aligned}\quad (117)$$

equation (116) takes the simpler form

$$\frac{d^2 x^i}{dt^2} = -c^2 g^{ii} \left(\frac{m}{r}\right)_{,i}. \quad (118)$$

For a central gravitational field of a mass M_G , we obtain the dynamic equation

$$\frac{d^2 r}{dt^2} = -\left(1 - \frac{2m}{r}\right) \frac{mc^2}{r^2} = -\left(1 - \frac{2GM_G}{c^2 r}\right) G \frac{M_G}{r^2}, \quad (119)$$

depending on the gravitational constant $G = 6.67259 \times 10^{-11} \text{ m}^3 \text{Kg}^{-1} \text{s}^{-2}$. From this expression, we obtain the Schwarzschild radius

$$r_0 = 2m = 2 \frac{G}{c^2} M_G, \quad (120)$$

where the gravitational acceleration falls to null, as a singularity of the Schwarzschild time-space interval,

$$ds^2 = \left(1 - \frac{r_0}{r}\right) c^2 dt^2 - \left(1 - \frac{r_0}{r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\varphi^2. \quad (121)$$

This singularity of a central gravitational field arises for a sufficiently large mass M_G , tending to confine this mass inside a sphere with a radius r_G smaller than the Schwarzschild radius (120), $r_G \leq r_0$, where the frequency of any emitted light

$$\frac{1}{\Delta t} = \sqrt{1 - \frac{r_0}{r}} \frac{c}{\Delta s} \quad (122)$$

is null. The spherical body with coordinates $r \leq r_0$ is a black hole, with a distribution of matter that will be found in the next section.

We consider a quantum particle in a radial motion in the gravitational field of a black hole,

$$\frac{dx^2}{ds} = \frac{dx^3}{ds} = 0, \quad (123)$$

as the time coordinate satisfies the geodesic equation

$$\begin{aligned} \frac{d^2 x^0}{ds^2} &= -\Gamma_{01}^0 \frac{dx^0}{ds} \frac{dx^1}{ds} - \Gamma_{10}^0 \frac{dx^1}{ds} \frac{dx^0}{ds} = -2g^{00} \Gamma_{001} \frac{dx^0}{ds} \frac{dx^1}{ds} \\ &= -g^{00} (g_{00,1} + g_{01,0} - g_{01,0}) \frac{dx^0}{ds} \frac{dx^1}{ds} = -g^{00} g_{00,1} \frac{dx^0}{ds} \frac{dx^1}{ds} \\ &= -g^{00} \frac{dg_{00}}{ds} \frac{dx^0}{ds}. \end{aligned} \quad (124)$$

With the contravariant metric elements (117),

$$g^{00} = \frac{1}{g_{00}}, \quad (125)$$

this equation takes the form

$$g_{00} \frac{d^2 x^0}{ds^2} + \frac{dg_{00}}{ds} \frac{dx^0}{ds} = 0, \quad (126)$$

which, by integration with a constant k_0 is

$$g_{00}v^0 = k_0. \quad (127)$$

From the fundamental equation (29) for this case,

$$g_{00}v^{0^2} + g_{11}v^{1^2} = 1, \quad (128)$$

with equation (127) we obtain

$$v^1 = \left(\frac{1 - k_0 v^0}{g_{11}} \right)^{1/2}. \quad (129)$$

For a particle approaching a black hole, with the relation

$$\frac{1}{g_{11}} = -g_{00} \quad (130)$$

between the Schwarzschild metric elements (117), and (127), from (129) we obtain the radial velocity

$$v^1 = -\left(k_0^2 - g_{00}\right)^{1/2}, \quad (131)$$

as the velocity of the time coordinate is

$$v^0 = \frac{k_0}{1 - \frac{2m}{r}}. \quad (132)$$

With these expressions, we obtain the radial velocity of a quantum particle in the gravitational field of a black hole,

$$\frac{dr}{dt} = c \frac{dx^1}{dx^0} = c \frac{v^1}{v^0} = -\frac{c}{k_0} \left(k_0^2 + \frac{2m}{r} - 1 \right)^{1/2} \left(1 - \frac{2m}{r} \right). \quad (133)$$

By this exact integration of the geodesic equation (126) for a relativistic velocity, with the integration constant $k_0 = 1$, we obtain a new correction factor for the approximate acceleration (119), asymptotically tending to this expression for $r \rightarrow \infty$

$$\frac{d^2r}{dt^2} = - \left[3 \left(1 - \frac{2m}{r} \right) - 2 \right] \left(1 - \frac{2m}{r} \right) \frac{mc^2}{r^2} \xrightarrow{r \rightarrow \infty} - \left(1 - \frac{2m}{r} \right) \frac{mc^2}{r^2}, \quad (134)$$

as the velocity (133) is

$$\frac{dr}{dt} = -c \left(1 - \frac{2m}{r} \right) \sqrt{\frac{2m}{r}}. \quad (135)$$

We notice that in the gravitational field of a black hole, an outer quantum particle, $2m < r < \infty$, approaches its Schwarzschild boundary, $r_0 = 2m$, but without passing this boundary, $\frac{dr}{dt} \xrightarrow{r \rightarrow r_0} 0$. From (134), we notice that for $r > 6m$ the quantum particle is accelerated towards the black hole, as at this point $r = 6m$, the particle is decelerated, and the velocity tends to null for $r = 2m$. At the same time, we notice that a particle inside a black hole, $r < 2m$, always has a positive velocity, but without passing the Schwarzschild boundary, when this velocity tends to null. This is because at the formation of the black hole, the matter in the centre of this black hole is thrown with a huge velocity, much larger than the light velocity $\frac{dr}{dt} \xrightarrow{r \rightarrow 0} \infty (> c)$.

Quantum particle in Universe: what is the Big Bang, what was before and after.

In the previous section we showed that, at the formation of a black hole, a particle in the central region, $r \ll 2m$, obtains a velocity (135) much larger than the light velocity, but with a deceleration (134), which leads to a null velocity – null acceleration at the Schwarzschild boundary $r = 2m$. It is interesting that, in a black hole, a quantum particle in an electromagnetic field with wave functions (103) and momentum (96),

$$\begin{aligned} \psi(x^i, t) &= \frac{1}{(2\pi\hbar)^{3/2}} \int \varphi(P^j, t) e^{\frac{i}{\hbar} \left[\left(\frac{-Mc g_{ij} \dot{x}^j}{\sqrt{g_{00} + g_{ik} \dot{x}^i \dot{x}^k}} + eA^j(x^\mu) \right) x^j + \left[eU(x^i) - eA^j(x^\mu) \dot{x}^j + Mc^2 \sqrt{g_{00} + g_{ij} \dot{x}^i \dot{x}^j} \right] t \right]} d^3P \\ \varphi(P^j, t) &= \frac{1}{(2\pi\hbar)^{3/2}} \int \psi(x^i, t) e^{-\frac{i}{\hbar} \left[\left(\frac{-Mc g_{ij} \dot{x}^j}{\sqrt{g_{00} + g_{ik} \dot{x}^i \dot{x}^k}} + eA^j(x^\mu) \right) x^j + \left[eU(x^i) - eA^j(x^\mu) \dot{x}^j + Mc^2 \sqrt{g_{00} + g_{ij} \dot{x}^i \dot{x}^j} \right] t \right]} d^3x, \end{aligned} \quad (136)$$

by a quantum decay $e\Delta U < 0$ in the electric potential $U(x^i)$, with the total energy and

momentum conservation generates a photon with a frequency $\omega = -\frac{e}{\hbar} \Delta U$ and a momentum $eA^i(x^\mu)$ propagating towards the inner part of the black hole, in the total momentum

$P^j = \frac{-Mc g_{ij} \dot{x}^j}{\sqrt{g_{00} + g_{ik} \dot{x}^i \dot{x}^k}} + eA^j(x^\mu)$. By the conservation of this total momentum in this process, the electromagnetic momentum $eA^i(x^\mu)$ determines a particle velocity leading to an increase in its

$$g_{ij} \dot{x}^i \dot{x}^j = g_{11} \dot{r}^2 = \frac{\dot{r}^2}{\frac{2m}{r} - 1}$$

kinetic term which, by increasing with the radius r , matters as a decrease in the transition and photon energies, $-e\Delta U = \hbar\omega$ – we obtain a redshift of the light emitted by a system approaching the Schwarzschild boundary, as a gravitational effect in the total gravitational field of the black hole.

We consider a quantum particle with a mass M as a distribution of matter described by an amplitude function/wave function of the form of a wave packet of the momentum eigenstates (26),

$$\begin{aligned}\psi(x^i, \tau) &= \frac{1}{(2\pi\hbar)^{3/2}} \int \varphi(P^j, \tau) e^{\frac{i}{\hbar}[P^j x^j - L(x^\alpha, v^\alpha)\tau]} d^3P \\ \varphi(P^j, \tau) &= \frac{1}{(2\pi\hbar)^{3/2}} \int \psi(x^i, \tau) e^{-\frac{i}{\hbar}[P^j x^j - L(x^\alpha, v^\alpha)\tau]} d^3x\end{aligned}\quad (137)$$

represented in the two conjugate spaces, of the coordinates (27),

$$(x^\alpha) = (x^0 = ct, x^1, x^2, x^3) = (x^0 = ct, x^i), \quad (138)$$

with a time-space interval (25),

$$ds \equiv c d\tau = \sqrt{g_{\alpha\beta} dx^\alpha dx^\beta} = \sqrt{g_{\alpha\beta} v^\alpha v^\beta} ds, \quad (139)$$

and momentum (30)

$$P^j = \frac{\partial L}{c \partial v^j} = -Mc g_{j\mu} v^\mu. \quad (140)$$

This momentum is a function of the relativistic Lagrangian (28),

$$L(x^\alpha, v^\alpha) = -Mc^2 \sqrt{g_{\alpha\beta} v^\alpha v^\beta}, \quad (141)$$

which depends on the velocities (24) in the proper time τ ,

$$v^\alpha = \frac{dx^\alpha}{ds} = \frac{dx^\alpha}{cd\tau}, \quad (142)$$

and the metric tensor $g_{\alpha\beta}$. The quantum dynamics of such a particle are described by the group velocities in the two spaces, (31) and (32)-(33), which are of the form of the Hamilton equations,

$$\begin{aligned}\frac{d}{d\tau} x^j &= \frac{\partial L}{\partial P_j} = c v^j \\ \frac{d}{d\tau} P_j &= \frac{\partial}{\partial x^j} L(x^\alpha, v^\alpha) = -\frac{1}{2} Mc^2 v^\alpha v^\beta g_{\alpha\beta,j}.\end{aligned}\quad (143)$$

We consider a central gravitational field of a matter system with spherical symmetry, described by the Schwarzschild metric elements (117),

$$g_{00} = 1 - \frac{2m}{r}, \quad g_{11} = -\left(1 - \frac{2m}{r}\right)^{-1}. \quad (144)$$

These metric elements have two singularities, $r = 0$, and $r = 2m$, proportional to the total mass M_G of this system with the Schwarzschild radius (120),

$$m = \frac{G}{c^2} M_G. \quad (145)$$

With the condition $k_0 = 1$ for the asymptotic condition (134), the time velocity (132) takes the explicit form

$$v^0 = \frac{dx^0}{ds} = \frac{1}{1 - \frac{2m}{r}} = -\frac{1}{\frac{2m}{r} - 1}, \quad (146)$$

as the radial velocity (131) with (144) is

$$v^1 = \frac{dr}{ds} = -\left(\frac{2m}{r}\right)^{1/2}. \quad (147)$$

Inside a black hole, with these expressions, in the time t of the reference system (138) considered for the whole matter system, we obtain a positive velocity of a particle,

$$\frac{dr}{dt} = c \frac{v^1}{v^0} = c \left(\frac{2m}{r} - 1\right) \left(\frac{2m}{r}\right)^{1/2} > 0, \quad r < 2m, \quad (148)$$

with a negative acceleration

$$\frac{d^2r}{dt^2} = -\left[3\left(\frac{2m}{r} - 1\right) + 2\right] \left(\frac{2m}{r} - 1\right) \frac{mc^2}{r^2} < 0, \quad r < 2m. \quad (149)$$

From velocities (146)-(148) we notice that inside a black hole the interval ds is negative. For $r > 6m$ a particle is attracted towards the black hole, as for $2m < r < 6m$ this particle is repelled, its velocity (148) and acceleration (149) takes null values at the Schwarzschild boundary $r = 2m$. Inside a black hole, $r < 2m$, a particle has a positive velocity, for $r < m$ much larger than the light velocity, being decelerated while it is approaching the Schwarzschild boundary $r = 2m$ - a matter distribution with a spherical symmetry tends to concentrate near the Schwarzschild boundary. Evidently, the matter of mass M_G , continuously accumulating around the Schwarzschild boundary from outside the black hole, leads to an increase in its radius (120). We notice that the Lagrangian (141) which, for radial motion in a system with spherical symmetry, is

$$L = -Mc^2 \sqrt{g_{00}v^{0^2} + g_{11}v^{1^2}}, \quad (150)$$

with metric elements (144) and velocities (146) and (147) is

$$L = -Mc^2 \sqrt{\left(1 - \frac{2m}{r}\right) \frac{1}{\left(1 - \frac{2m}{r}\right)^2} - \left(1 - \frac{2m}{r}\right)^{-1} \frac{2m}{r}} = -Mc^2 \quad (151)$$

With the radial momentum (140) with (144) and (147),

$$P = P^1 = \frac{\partial L}{c \partial v^1} = -Mc g_{11} v^1 = -Mc \left(1 - \frac{2m}{r}\right)^{-1} \left(\frac{2m}{r}\right)^{1/2} = \frac{Mc}{\frac{2m}{r} - 1} \left(\frac{2m}{r}\right)^{1/2} \quad (152)$$

the Lagrangian (150) with the second term as a function of this momentum, takes an explicit form for the wave functions (137) as functions of coordinates and momentum,

$$L = -Mc^2 \sqrt{\left(\frac{2m}{r} - 1\right) \frac{P^2}{M^2 c^2} - \frac{1}{\frac{2m}{r} - 1}} = -Mc^2 \quad (153)$$

From the group velocity of the wave function (137) with momentum (152) and Lagrangian (153), we obtain the radial velocity of a quantum particle in a black hole, in agreement with (148),

$$\frac{dr}{dt} = c \frac{v^1}{v^0} = \frac{1}{v^0} \frac{dr}{d\tau} = \frac{1}{v^0} \frac{\partial L}{\partial P} = c \left(\frac{2m}{r} - 1\right) \left(\frac{2m}{r}\right)^{1/2} \quad (154)$$

This means that a quantum particle in a black hole that travels from the centre towards the Schwarzschild boundary, in the approximation of a time constant gravitational field, cannot pass this boundary. We notice that the momentum (152) of a quantum particle, which is

$$P = \frac{Mc}{\left(\frac{2m}{r}\right)^{1/2} - \left(\frac{r}{2m}\right)^{1/2}}, \quad (155)$$

takes the null value in the centre $r = 0$, which means that in this region a particle with wave functions of the form (137) is crushed up to a large spreading by the gravitational field. At the Schwarzschild boundary $r = 2m$, the particle obtains momentum tending to infinity. When the particle approaches this boundary from the inside or the outside of the black hole, although its velocity (147) tends to zero, the momentum tends to infinity as the mass increases to infinity, which dominates the velocity decrease to null.

Obviously, these singularity regions arise only in a matter system with a spherical symmetry, which is not conceivable in a cosmic system of bodies of different dimensions and velocities. However in a sufficiently large agglomeration of matter, the quantum particles are crushed up to a homogeneous distribution of matter, and the gravitational field takes a form with spherical symmetry. The density $\rho_G(r)$ of this matter is described by Einstein's gravitation law²,

$$R = 8\pi\rho = 8\pi \frac{G}{c^2} \rho_G, \quad (156)$$

for the scalar curvature

$$R = g^{\mu\nu} R_{\mu\nu} = g^{00} R_{00} + g^{11} R_{11}, \quad (157)$$

as a function of the contravariant metric elements (117),

$$g^{00} = \left(1 - \frac{2m}{r}\right)^{-1}, \quad g^{11} = -\left(1 - \frac{2m}{r}\right). \quad (158)$$

With the Ricci tensor elements as functions of the metric elements², the two terms of the scalar curvature (157) are

$$\begin{aligned} R_{00} &= \frac{1}{2} g^{\alpha\beta} (g_{\beta\alpha,00} - g_{\beta 0,0\alpha} + g_{00,\beta\alpha} - g_{0\alpha,\beta 0}) = \frac{1}{2} g^{11} (g_{11,00} + g_{00,11}) \\ R_{11} &= \frac{1}{2} g^{\alpha\beta} (g_{\beta\alpha,11} - g_{\beta 1,1\alpha} + g_{11,\beta\alpha} - g_{1\alpha,\beta 1}) = \frac{1}{2} g^{00} (g_{00,11} + g_{11,00}). \end{aligned} \quad (159)$$

With the metric elements (158), we obtain

$$R = g^{00} \frac{1}{2} g^{11} (g_{11,00} + g_{00,11}) + g^{11} \frac{1}{2} g^{00} (g_{00,11} + g_{11,00}) = -g_{00,11} - g_{11,00}. \quad (160)$$

From the metric elements (144),

$$g_{00,11} = -2 \frac{2m}{r^3}, \quad (161)$$

as

$$g_{11,00} = -2 \left(1 - \frac{2m}{r}\right)^{-3} \frac{2m}{r^3} \left(\frac{dr}{dx^0}\right)^2 + \left(1 - \frac{2m}{r}\right)^{-2} \left(\frac{2m}{r^2}\right) \frac{d^2 r}{dx^{0^2}}. \quad (162)$$

With the two time derivatives (148) and (149), which are

$$\begin{aligned} \frac{dr}{dx^0} &= -\left(1 - \frac{2m}{r}\right) \left(\frac{2m}{r}\right)^{1/2} \\ \frac{d^2 r}{dx^{0^2}} &= -\frac{d}{dr} \left[\left(1 - \frac{2m}{r}\right) \left(\frac{2m}{r}\right)^{1/2} \right] \frac{dr}{dx^0} = -\left[3 \left(1 - \frac{2m}{r}\right) - 2 \right] \left(1 - \frac{2m}{r}\right) \frac{m}{r^2}, \end{aligned} \quad (163)$$

we obtain

$$g_{11,00} = -2 \left(3 + \frac{2}{1 - \frac{2m}{r}} \right) \left(\frac{m}{r^2} \right)^2. \quad (164)$$

With (161) and (164), the scalar curvature (160) is

$$R = \left[2 + \frac{m}{r} \left(3 + \frac{2}{1 - \frac{2m}{r}} \right) \right] \frac{2m}{r^3} \quad (165)$$

From (145) with Einstein's gravitation law (156), we obtain an integral equation for the scalar curvature, as a function of the gravitational field of total matter with mass M_G with a spherical symmetry, occupying a spherical domain (r_1, r_2) .

$$\begin{aligned} m &= \frac{G}{c^2} M_G = \frac{G}{c^2} 4\pi \int_{r_1}^{r_2} \rho_G r^2 dr = 4\pi \int_{r_1}^{r_2} \rho r^2 dr \\ &= \frac{1}{2} \int_{r_1}^{r_2} R r^2 dr \end{aligned} \quad (166)$$

With (165), this equation is

$$\int_{r_1}^{r_2} \left[1 + \frac{m}{r} \left(\frac{3}{2} + \frac{1}{1 - \frac{2m}{r}} \right) \right] \frac{2}{r} dr = 1 \quad (167)$$

By the integration of this function,

$$\left[1 + \frac{m}{r} \left(\frac{3}{2} + \frac{1}{1 - \frac{2m}{r}} \right) \right] \frac{2}{r} = \frac{2r - 2m}{r^2 - 2mr} + \frac{3m}{r^2} \quad (168)$$

we obtain the equation

$$\ln \left| r(r - 2m) \right| \Big|_{r_1}^{r_2} - \frac{3m}{r} \Big|_{r_1}^{r_2} = 1 \quad (169)$$

which, with the notation $x_2 = \frac{r_2}{2m}$ for the normalized higher matter bound as a function of the

normalized lower bound $x_1 = \frac{r_1}{2m}$, and the parameter

$$\eta = \frac{x_2(x_2 - 1)}{x_1(1 - x_1)} \quad (170)$$

takes the form

$$\ln \eta + \frac{3}{2} \left(\frac{1}{x_1} - \frac{1}{x_2} \right) = 1 \quad (171)$$

From (170) and (171), we obtain

$$x_2 = \frac{1}{2} + \sqrt{\frac{1}{4} + \eta(x_1 - x_1^2)} \quad (172)$$

with the particular values for the entire time evolution of the matter distribution from the formation of its spherical symmetry, $x_1 = 0$, up to the total matter concentration at the Schwarzschild boundary $x_1 = 1$,

$$\begin{cases} x_1 = 0 \rightarrow x_2 = 1 \\ x_1 = \frac{1}{2} \rightarrow \ln \eta = -\frac{2\sqrt{1+\eta}-1}{\sqrt{1+\eta}+1}, \quad \eta = 0.52, \quad x_2 = \frac{1}{2}(1+\sqrt{1+\eta}) = 1.116 \\ x_1 = 1 \rightarrow x_2 = 1. \end{cases} \quad (173)$$

This time evolution, in the gravitational field of the black hole total matter, we call total field, is described by equation (148), which is

$$dx^0 = \left(\frac{2m}{r} - 1\right)^{-1} \left(\frac{2m}{r}\right)^{-1/2} dr. \quad (174)$$

With the notations

$$x = \frac{r}{2m}, \quad dr = 2m dx, \quad x = y^2, \quad dx = 2y dy, \quad (175)$$

this equation takes the form

$$dx^0 = 2m \frac{x^{3/2}}{1-x} dx = 4m \left(\frac{1}{1-y^2} - 1 - y^2 \right) dy. \quad (176)$$

We obtain the time evolution

$$\begin{aligned} x^0(r) &= 4m \left\{ \frac{1}{2} [\ln(1+y) - \ln(1-y)] - y - \frac{1}{3} y^3 \right\} \\ &= 4m \left\{ \frac{1}{2} \left[\ln \left(1 + \sqrt{\frac{r}{2m}} \right) - \ln \left(1 - \sqrt{\frac{r}{2m}} \right) \right] - \sqrt{\frac{r}{2m}} - \frac{1}{3} \left(\frac{r}{2m} \right)^{3/2} \right\}. \end{aligned} \quad (177)$$

From the second term, we notice that a quantum particle reaches the Schwarzschild boundary only in an infinite time, of course, in the approximation of a time constant gravitational field, when this boundary cannot be passed. Otherwise, by gravitational fluctuations, evaporation and absorption processes arise through the boundary of the black hole. For a very short time from the formation of the matter distribution with spherical symmetry, leading to this time evolution, nonzero terms are obtained only in the fifth-order approximation of a Taylor series expansion.

$$\begin{aligned}
 x^0(r) = ct(r) &= 4m \left\{ \frac{1}{2} \left[\begin{aligned} &\cancel{x} - \frac{1}{2} \cancel{y} + \frac{1}{3} \cancel{y^3} - \frac{1}{4} \cancel{y^4} + \frac{1}{5} y^5 + \dots \\ &- \left(-\cancel{x} - \frac{1}{2} \cancel{y} - \frac{1}{3} \cancel{y^3} - \frac{1}{4} \cancel{y^4} - \frac{1}{5} y^5 + \dots \right) \end{aligned} \right] - \cancel{x} - \frac{1}{3} \cancel{y^3} \right\} \\
 &= \frac{4m}{5} y^5 + \dots = \frac{4m}{5} \left(\frac{r}{2m} \right)^{5/2} + \dots = \frac{2}{5} \left(\frac{r}{2m} \right)^{3/2} r + \dots
 \end{aligned}
 \tag{178}$$

This means an initial velocity

$$\frac{r}{t} = c \frac{5}{2} \left(\frac{2m}{r} \right)^{3/2} \xrightarrow{r \rightarrow 0} \infty
 \tag{179}$$

much larger than the light velocity, i.e. an explosion of the matter at the symmetry centre, and a rapid inflation of this matter and of the matter around. This suggests that our Universe is a huge black hole, with this explosion as the Big Bang-inflation process. For our universe¹⁸, with

a supposed radius of $2m = 4.3993 \times 10^{26} \text{ m}$ and the mass (145), $M_G = \frac{c^2}{G} m = 2.9628 \times 10^{53} \text{ Kg}$

with the gravitational constant $G = 6.67259 \times 10^{-11} \text{ m}^3 \text{ s}^{-2} \text{ Kg}^{-1}$ and the light velocity $c = 299792458 \text{ ms}^{-1}$, this process consists of the matter travelling with the velocities (148) towards the Schwarzschild boundary. Obviously, most of this large mass is situated around the Schwarzschild boundary of our universe, as the mass of the black and visible matter we observe in the Universe from our planet is significantly smaller. Compared to the standard Big Bang-inflation concept of the standard theory, of an explosion of the matter of the Universe initially concentrated in a single point, and continuously expanding to infinity, in the framework of the general theory of relativity we found an explosion only of the matter at the centre and around this centre, of a sufficiently large matter concentration with spherical symmetry, and an expansion of this matter with a velocity much larger than the light velocity, a dynamic we also call big bang-inflation. This process has the tendency to empty the Universe, sending its inner matter towards its Schwarzschild boundary, but by the violent collisions of bodies with the very different velocities (148), the remaining inner cosmic bodies have been scattered throughout the whole Universe. In the black hole of our universe with a radius $2m = 4.65 \times 10^9 \text{ ly} = 4.4 \times 10^{26} \text{ m}$, a local reference system situated at half of this radius, m , has an

acceleration (149) $\frac{d^2 r}{dt^2} = -5 \frac{c^2}{r} \approx -10^{-9} \text{ ms}^{-2}$, i.e. it is a quasi-inertial system. An expansion of our

universe could be understood as an increase of its radius $r_0 = 2m = 2 \frac{G}{c^2} M_G$ by the increase of its mass M_G on the account of the matter attracted from the Total Universe, where this huge black hole, which is our Universe, has been formed.

QUANTUM PARTICLE IN A GRAVITATIONAL WAVE AND THE GRAVITON SPIN

We consider a gravitational wave described by the D'Alembert equation (107), usually sufficiently weak for inducing only a small velocity to a quantum particle,

$$\dot{x}^i \ll \dot{x}^0 \simeq 1, \quad (180)$$

which means a time-space interval approximately equal to the time interval, $ds \simeq dx^0$, $d\tau \simeq dt$. In this case, the quantum dynamics of a particle with wave functions (56),

$$\begin{aligned} \psi(x^i, t) &= \frac{1}{(2\pi\hbar)^{3/2}} \int \varphi(P^j, t) e^{\frac{i}{\hbar} [P^j x^j - L(x^\alpha, v^\alpha)_t]} d^3 P \\ \varphi(P^j, t) &= \frac{1}{(2\pi\hbar)^{3/2}} \int \psi(x^i, t) e^{-\frac{i}{\hbar} [P^j x^j - L(x^\alpha, v^\alpha)_t]} d^3 x, \end{aligned} \quad (181)$$

is described by the Lagrangian (28)

$$L(x^\alpha, \dot{x}^\alpha) = -Mc^2 \sqrt{g_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta}, \quad (182)$$

and the momentum

$$P^j = \frac{\partial L}{\partial \dot{x}^j} = -Mc g_{ij} \dot{x}^i, \quad (183)$$

as the fundamental equation (29) is

$$g_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta = 1. \quad (184)$$

With this equation, from the group velocity in the momentum space of the wave packet (181) with the Lagrangian (182),

$$\begin{aligned} \frac{d}{dt} P^j = c \dot{P}^j &= -Mc^2 g_{ij,k} \dot{x}^k \dot{x}^i - Mc^2 g_{ij} \ddot{x}^i = \frac{\partial L}{\partial x^j} \\ &= -Mc^2 \frac{\dot{x}^\alpha \dot{x}^\beta g_{\alpha\beta,j}}{2\sqrt{g_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta}} = -\frac{1}{2} Mc^2 g_{\alpha\beta,j} \dot{x}^\alpha \dot{x}^\beta, \end{aligned} \quad (185)$$

in the realistic case (180) of a slow velocity, we obtain the simpler expression of the particle acceleration, proportional to the gradient of this potential,

$$\ddot{x}^k = \frac{1}{2} g^{kj} g_{00,j}. \quad (186)$$

At the same time, for the gravitational wave equation (107) we consider a first-order solution²,

$$g_{\rho\sigma} = u_{\rho\sigma} l_\mu x^\mu, \quad (187)$$

with the amplitude tensor

$$u_{\rho\sigma} = u g_{\rho\sigma}, \quad u^{\rho\sigma} = u g^{\rho\sigma} \quad (188)$$

proportional to the scalar amplitude u , and the polarization tensor l^μ with the normalization condition

$$l^\nu l_\nu = g^{\mu\nu} l_\mu l_\nu = 0 \quad (189)$$

By multiplying the second equation (188) with the covariant metric tensor

$$g_{\mu\rho} u^{\rho\sigma} = u g_{\mu\rho} g^{\rho\sigma}, \quad (190)$$

we obtain the elements of the amplitude tensor as functions of the scalar amplitude,

$$u^\sigma_\mu = u \delta^\sigma_\mu. \quad (191)$$

By multiplying this expression with the polarization vector, for these elements we obtain a system of equations,

$$l_\sigma u^\sigma_\mu = l_\mu u, \quad (192)$$

as by the contraction of the two indexes we obtain the trace of the amplitude matrix,

$$u^\mu_\mu = 4u. \quad (193)$$

With the expression (187) of the metric tensor, the dynamic equation (186) takes the form of a constant acceleration in the propagation direction l^k of the gravitational wave,

$$\ddot{x}^k = \frac{1}{2} g^{kj} u_{00} l_j = \frac{1}{2} u_{00} l^k. \quad (194)$$

For the realistic case of a weak gravitational wave,

$$\begin{aligned} g_{00} &= 1, \quad g_{11} = -1, \quad g_{22} = -1, \quad g_{33} = -1 \\ g^{00} &= 1, \quad g^{11} = -1, \quad g^{22} = -1, \quad g^{33} = -1, \end{aligned} \quad (195)$$

with the amplitude element (188)

$$u_{00} = u g_{00} = u, \quad (196)$$

the dynamic equation (194) takes the form of a constant acceleration proportional to the scalar amplitude of the gravitational wave,

$$\ddot{x}^k = \frac{1}{2} u l^k. \quad (197)$$

For a gravitational wave with the metric tensor (195), propagating in the direction x^3 of the polarization vector

$$\begin{aligned} l_0 &= 1, \quad l_1 = l_2 = 0, \quad l_3 = -1 \\ l^0 &= 1, \quad l^1 = l^2 = 0, \quad l^3 = 1, \end{aligned} \quad (198)$$

which satisfies the normalization condition (189), the amplitude equations (192) take the form

$$\begin{aligned}
 u_0^0 - u_0^3 &= g^{00}u_{00} - g^{33}u_{30} = u_{00} + u_{30} = u \\
 u_1^0 - u_1^3 &= g^{00}u_{01} - g^{33}u_{31} = u_{01} + u_{31} = 0 \\
 u_2^0 - u_2^3 &= g^{00}u_{02} - g^{33}u_{32} = u_{02} + u_{32} = 0 \\
 u_3^0 - u_3^3 &= g^{00}u_{03} - g^{33}u_{33} = u_{03} + u_{33} = -u.
 \end{aligned} \tag{199}$$

With the symmetry property of the amplitude tensor according to (187), and the metric tensor (195), from these equations and the first expression (188), we obtain

$$\begin{aligned}
 u_{00} + u_{33} &= -2u_{03} \\
 u_{00} - u_{33} &= 2u = -(u_{11} + u_{22}) \\
 u_{10}^2 &= u_{13}^2 \\
 u_{20}^2 &= u_{23}^2.
 \end{aligned} \tag{200}$$

With these equations and the contravariant amplitude tensor

$$u^{\alpha\beta} = g^{\alpha\mu}g^{\beta\nu}u_{\mu\nu} = g^{\alpha\alpha}g^{\beta\beta}u_{\alpha\beta}, \tag{201}$$

we obtain the scalar

$$\begin{aligned}
 I_0 &= u_{\alpha\beta}u^{\alpha\beta} - 4u^2 = u_{\alpha\beta}u^{\alpha\beta} - 2u^2 - 2u^2 \\
 &= u_{00}^2 + u_{11}^2 + u_{22}^2 + u_{33}^2 - \cancel{2u_{10}^2} - \cancel{2u_{20}^2} - 2u_{03}^2 + 2u_{12}^2 + \cancel{2u_{23}^2} + \cancel{2u_{13}^2} \\
 &\quad - \frac{1}{2}(u_{00} - u_{33})^2 - 2u^2 \\
 &= u_{11}^2 + u_{22}^2 + 2u_{12}^2 + u_{00}^2 + u_{33}^2 - \frac{1}{2}(u_{00} + u_{33})^2 - \frac{1}{2}(u_{00} - u_{33})^2 - 2u^2 \\
 &= u_{11}^2 + u_{22}^2 + 2u_{12}^2 - \frac{1}{2}(u_{11} + u_{22})^2 \\
 &= \frac{1}{2}(u_{11} - u_{22})^2 + 2u_{12}^2
 \end{aligned} \tag{202}$$

as an invariant of a rotation of the amplitude tensor in the plane (x^1, x^2) , perpendicular to the direction x^3 of propagation of the gravitational wave. For the rotation operator R with an angle

$-\frac{\pi}{2}$ of a vector $A = (A^1, A^2)$ in this plane,

$$RA = (RA^1, RA^2) = (A^2, -A^1),$$

(203)

we obtain the equation

$$R^2 A^1 = -A^1, \tag{204}$$

with the eigenvalue $R = \pm i$. With these expressions, the rotation with an angle $-\frac{\pi}{2}$ of a second-order symmetric tensor is

$$\begin{aligned} Ru_{11} &= u_{12} = u_{21} = \frac{1}{2}(u_{12} + u_{21}) \\ Ru_{22} &= -u_{21} = -u_{12} = -\frac{1}{2}(u_{12} + u_{21}) \\ Ru_{21} &= Ru_{12} = u_{22} = -u_{11} = \frac{1}{2}(u_{22} - u_{11}). \end{aligned} \quad (205)$$

From the first two equations we obtain

$$\begin{aligned} R(u_{11} + u_{22}) &= 0 \\ R(u_{11} - u_{22}) &= 2u_{12}. \end{aligned} \quad (206)$$

At the same time, from equation (205) we obtain the equation

$$R^2(u_{11} - u_{22}) = R(u_{12} + u_{21}) = u_{22} - u_{11}, \quad (207)$$

with the eigenvalue $R = \pm i$ of a vector rotation. With the second equation (206), we find that with a rotation R , the first term of the invariant (202) takes the form of the second term, and conversely, the second term takes the form of the first term,

$$\begin{aligned} \frac{1}{2}[R(u_{11} - u_{22})]^2 &= \frac{1}{2}(2u_{12})^2 = 2u_{12}^2 \\ 2(Ru_{12})^2 &= 2\left[\frac{1}{2}(u_{22} - u_{11})\right]^2 = \frac{1}{2}(u_{11} - u_{22})^2. \end{aligned} \quad (208)$$

From these expressions, we notice that the two terms of the invariant (202) change one another

by a rotation with an angle $-\frac{\pi}{2}$, i.e. every term retakes its value by a rotation with an angle $-\pi$, which means that by a complete rotation with -2π , the terms of the invariant (202) retake their values two times - in the following we shall see that this variation corresponds to a spin 2.

We consider the rotation operator with an arbitrary angle $\delta\vec{\alpha}$, $R^{\delta\vec{\alpha}} = e^{i\vec{S}\delta\vec{\alpha}}$, as a function of the angular momentum

$$\vec{S} = -i\vec{r} \times \frac{\partial}{\partial \vec{r}}. \quad (209)$$

For a rotation with an angle $-\pi$ of a vector $A^i(\vec{r})$,

$$R^{-\pi} A^i(\vec{r}) = e^{-iS\pi} A^i(\vec{r}) = -A^i(\vec{r}), \quad (210)$$

we obtain the angular momentum of a vector $S = 1$, which we call spin. From the rotation with an arbitrary angle $\delta\vec{\alpha}$ of the product of the amplitude tensor with two vectors, as a scalar, is invariant for any rotation,

$$\begin{aligned} \text{Scalar} &= u_{ij}(x, y) A^i(x, y) B^j(x, y) = [R^{\delta\vec{\alpha}} u_{ij}(x, y)] [R^{\delta\vec{\alpha}} A^i(x, y)] [R^{\delta\vec{\alpha}} B^j(x, y)] \\ &= [e^{i\vec{S}_t \delta\vec{\alpha}} u_{ij}(x, y)] [e^{i\vec{S} \delta\vec{\alpha}} A^i(x, y)] [e^{i\vec{S} \delta\vec{\alpha}} B^j(x, y)] \\ &= [e^{i\vec{S}_t \delta\vec{\alpha}} u_{ij}(x, y)] [e^{i\delta\vec{\alpha}} A^i(x, y)] [e^{i\delta\vec{\alpha}} B^j(x, y)], \end{aligned} \quad (211)$$

for this tensor we obtain the spin $S_t = 2$, we call the graviton spin. This spin describes a rotation in the plane (x^1, x^2) of the metric tensor in a gravitational wave propagating in the direction x^3 . From the fundamental equation (184), we notice that this rotation is correlated with a rotation of the velocity tensor $\dot{x}^\alpha \dot{x}^\beta$, in the momentum space.

We notice that fundamental invariant (184) of a quantum particle under the action of a gravitational wave with the metric tensor (187), is composed of two invariant factors

$$g_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta = u_{\alpha\beta} l_\mu x^\mu \dot{x}^\alpha \dot{x}^\beta = I_G I_C = 1, \quad (212)$$

the gravitational invariant, depending on the amplitude of the metric tensor and the matter velocity field,

$$I_G = u_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta = u, \quad (213)$$

and the coordinate invariant, depending on the polarization of the gravitational wave and the coordinates,

$$I_C = l_\mu x^\mu = \frac{1}{u}. \quad (214)$$

The gravitational invariant (213) describes a rotation with the invariant (202) correlated to the matter velocity, with the spin $S_t = 2$, called the graviton spin. The coordinate invariant (214) describes a matter rotation, correlated with the coordinate field, with the invariant (189), which with (195) takes the form

$$l_1^2 + l_2^2 = (-Rl_2)^2 + (Rl_1)^2 = (Rl_2)^2 + (Rl_1)^2 = l_0^2 + l_3^2, \quad (215)$$

describing a rotation of the polarization vector in the plane (x^1, x^2) perpendicular to the direction of the axis x^3 of the particle motion. From a rotation with 2π , $R = e^{i2\pi S}$, we notice that this relation is valid for $R = -1$, describing Fermions, with a half-integer spin, and $R = 1$, describing Bosons, with an integer spin.

For the gravitational wave equation (107), we can consider a second-order solution

$$g_{\rho\sigma} = u_{\rho\sigma} l_{\mu\nu} x^\mu x^\nu, \quad (216)$$

with an amplitude tensor $u_{\rho\sigma}$ and a polarization tensor $l_{\mu\nu}$ which, according to this equation, satisfies the normalization equation

$$l_\nu^\nu = 0. \quad (217)$$

From (186), we obtain the dynamic equation

$$\ddot{x}^k = \frac{1}{2} g^{kj} g_{00,j} = \frac{1}{2} g^{kj} u_{00} l_{\mu j} x^\mu = \frac{1}{2} u_{00} l_\mu^k x^\mu. \quad (218)$$

For a gravitational field oscillating in the direction x^1 ,

$$l_0^0 = 1, \quad l_1^1 = -1, \quad l_2^2 = l_3^3 = 0, \quad (219)$$

which satisfies the normalization condition (217), with (196) the dynamic equation (218) takes the form of a harmonic oscillator of a quantum particle in a gravitational wave with a scalar amplitude u ,

$$\ddot{x}^1 = -\frac{1}{2} u x^1. \quad (220)$$

In this way, we found that a quantum particle is a complex entity, as an element of the 4-dimensional curved space in a gravitational wave, with a spin of 2 and an element of a distribution of mater in this space, with a half integer spin for Fermions, and an integer spin for Bosons.

RELATIVISTIC PROPAGATION WAVE FUNCTIONS

A relativistic quantum particle is usually described by a Schrödinger equation with the relativistic Dirac's Hamiltonian^{5,6,15,16,17}, usually called the Schrödinger-Dirac equation. However, we consider propagation states of the form (56), as wave packets in the two conjugate spaces of the coordinates and momentum,

$$\begin{aligned}
\psi(\vec{r}, t) &= \langle \vec{r}, t | \psi \rangle = \frac{1}{(2\pi\hbar)^{3/2}} \int \varphi(\vec{p}, t) e^{\frac{i}{\hbar} [\vec{p}\vec{r} - L(\vec{r}, \vec{p})] t} d^3\vec{p} \\
&= \frac{1}{(2\pi\hbar)^{3/2}} \int \varphi(\vec{p}, t) e^{\frac{i}{\hbar} \{ \vec{p}\vec{r} - [\vec{p}\vec{r} - H(\vec{p}, \vec{r})] t \}} d^3\vec{p} = \int \langle \vec{r}, t | \vec{p}, t \rangle \langle \vec{p}, t | \psi \rangle d^3\vec{p} \\
&= e^{\frac{i}{\hbar} \vec{p}\vec{r}} \frac{1}{(2\pi\hbar)^{3/2}} \int \varphi(\vec{p}, t) e^{-\frac{i}{\hbar} [\vec{p}\vec{r} - H(\vec{p}, \vec{r})] t} d^3\vec{p} = e^{\frac{i}{\hbar} \vec{p}\vec{r}} \psi_t(\vec{r}, t) \\
\varphi(\vec{p}, t) &= \langle \vec{p}, t | \varphi \rangle = \frac{1}{(2\pi\hbar)^{3/2}} \int \psi(\vec{r}, t) e^{-\frac{i}{\hbar} [\vec{p}\vec{r} - L(\vec{r}, \vec{p})] t} d^3\vec{r} \\
&= \frac{1}{(2\pi\hbar)^{3/2}} \int \psi(\vec{r}, t) e^{-\frac{i}{\hbar} \{ \vec{p}\vec{r} - [\vec{p}\vec{r} - H(\vec{p}, \vec{r})] t \}} d^3\vec{r} = \int \langle \vec{p}, t | \vec{r}, t \rangle \langle \vec{r}, t | \psi \rangle d^3\vec{r} \\
&= e^{-\frac{i}{\hbar} \vec{p}\vec{r}} \frac{1}{(2\pi\hbar)^{3/2}} \int \psi(\vec{r}, t) e^{\frac{i}{\hbar} [\vec{p}\vec{r} - H(\vec{p}, \vec{r})] t} d^3\vec{r} = e^{-\frac{i}{\hbar} \vec{p}\vec{r}} \varphi_t(\vec{p}, t),
\end{aligned} \tag{221}$$

as solutions of the relativistic quantum dynamic equations

$$\begin{aligned}
i\hbar \frac{\partial}{\partial t} \psi_t(\vec{r}, t) &= [\vec{p}\vec{r} - H(\vec{p}, \vec{r})] \psi_t(\vec{r}, t) \\
i\hbar \frac{\partial}{\partial t} \varphi_t(\vec{p}, t) &= -[\vec{p}\vec{r} - H(\vec{p}, \vec{r})] \varphi_t(\vec{p}, t).
\end{aligned} \tag{222}$$

The wave functions (221) satisfy the relativistic quantum principle of the invariance of the time dependent phase variation (28), $L(\vec{r}, \vec{r}) dt = -Mc ds$, which is equivalent to the relativistic principle of the invariance of the time-space interval. In these wave functions we distinguish the momentum and coordinate eigenfunctions,

$$\begin{aligned}
\langle \vec{r}, t | \vec{p}, t \rangle &= \frac{1}{(2\pi\hbar)^{3/2}} e^{\frac{i}{\hbar} \{ \vec{p}\vec{r} - [\vec{p}\vec{r} - H(\vec{p}, \vec{r})] t \}} \\
\langle \vec{p}, t | \vec{r}, t \rangle &= \frac{1}{(2\pi\hbar)^{3/2}} e^{-\frac{i}{\hbar} \{ \vec{p}\vec{r} - [\vec{p}\vec{r} - H(\vec{p}, \vec{r})] t \}}.
\end{aligned} \tag{223}$$

For a free quantum particle with the Hamiltonian⁶

$$H = c\sqrt{M^2 c^2 + \vec{p}^2} = cE, \tag{224}$$

we consider the scalar Mc as an invariant of a generalized momentum P_μ ,

$$Mc\hat{1} = \sqrt{\left(\frac{H}{c}\right)^2 - \vec{p}^2} \hat{1} = \sqrt{p^\mu p_\mu} \hat{1} = \sqrt{g^{\mu\nu} p_\mu p_\nu} \hat{1} = \gamma^\mu p_\mu, \tag{225}$$

where $\hat{1}$ is the four-dimensional unity matrix, and γ^μ are matrices that satisfy the Clifford algebra. With the Pauli spin matrices σ^i , these matrices are

$$\gamma^0 = \begin{pmatrix} \hat{1} & \hat{0} \\ \hat{0} & -\hat{1} \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} \hat{0} & \sigma_i \\ -\sigma_i & \hat{0} \end{pmatrix}, \quad (226)$$

where $\hat{1}$ is the two-dimensional unity matrix, and $\hat{0}$ is the two-dimensional null matrix. In a flat space, $g^{00} = g_{00} = 1$, $g^{11} = g_{11} = -1$, $g^{22} = g_{22} = -1$, $g^{33} = g_{33} = -1$, we consider the wave equations (221) as functions of Dirac's Hamiltonian², for the covariant and contravariant momentum four-vectors

$$\begin{aligned} H\hat{1} &= c\sqrt{M^2c^2 + \vec{p}^2}\hat{1} = c(\alpha_0Mc + \alpha_1p_1 + \alpha_2p_2 + \alpha_3p_3) \\ &= c(\alpha_0Mc + \alpha_1p^1 + \alpha_2p^2 + \alpha_3p^3), \end{aligned} \quad (227)$$

as operators in the two dynamic equations (222), of the coordinate wave function and, respectively, of the momentum wave function, which depend on Dirac's matrices

$$\alpha_0 = \begin{pmatrix} \hat{1} & \hat{0} \\ \hat{0} & -\hat{1} \end{pmatrix}, \quad \alpha_i = \begin{pmatrix} \hat{0} & \sigma_i \\ \sigma_i & \hat{0} \end{pmatrix}. \quad (228)$$

With these expressions, from (222) we obtain the dynamic equations

$$\begin{aligned} i\hbar \frac{\partial}{\partial t} \psi_t(\vec{r}, t) \hat{1} &= \left[\vec{p} \cdot \vec{\hat{r}} \hat{1} - c(\alpha_0Mc + \alpha_1p_1 + \alpha_2p_2 + \alpha_3p_3) \right] \psi_t(\vec{r}, t) \\ i\hbar \frac{\partial}{\partial t} \varphi_t(\vec{p}, t) \hat{1} &= - \left[\vec{p} \cdot \vec{\hat{r}} \hat{1} - c(\alpha_0Mc + \alpha_1p^1 + \alpha_2p^2 + \alpha_3p^3) \right] \varphi_t(\vec{p}, t), \end{aligned} \quad (229)$$

which, with the momentum elements

$$p_0 = p^0 = i\hbar \frac{\partial}{\partial x^0} = i\hbar \frac{\partial}{c\partial t}, \quad p_i = -p^i = i\hbar \frac{\partial}{\partial x^i}, \quad (230)$$

are

$$\begin{aligned} &\left[i\hbar \left(\hat{1} \frac{\partial}{\partial x^0} + \alpha_1 \frac{\partial}{\partial x^1} + \alpha_2 \frac{\partial}{\partial x^2} + \alpha_3 \frac{\partial}{\partial x^3} \right) + \alpha_0Mc - \vec{p} \cdot \frac{\vec{\hat{r}}}{c} \hat{1} \right] \psi_t(\vec{r}, t) = 0 \\ &\left(p^0 - \alpha_1p^1 - \alpha_2p^2 - \alpha_3p^3 - \alpha_0Mc + \vec{p} \cdot \frac{\vec{\hat{r}}}{c} \hat{1} \right) \varphi_t(\vec{p}, t) = 0. \end{aligned} \quad (231)$$

By multiplication with $\alpha_0 = \gamma^0$, the coefficients of these equations take the form of the Clifford algebra elements,

$$\alpha_0 \alpha_i = \begin{pmatrix} \hat{1} & \hat{0} \\ \hat{0} & -\hat{1} \end{pmatrix} \begin{pmatrix} \hat{0} & \sigma_i \\ \sigma_i & \hat{0} \end{pmatrix} = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} = \gamma^i \quad (232)$$

With (230) and (232), the dynamic equations (231) become

$$\left[i\hbar \left(\gamma^0 \frac{\partial}{\partial x^0} + \gamma^1 \frac{\partial}{\partial x^1} + \gamma^2 \frac{\partial}{\partial x^2} + \gamma^3 \frac{\partial}{\partial x^3} \right) + \hat{1}Mc - \gamma^0 \vec{p} \frac{\dot{\vec{r}}}{c} \right] \psi_t(\vec{r}, t) = 0$$

$$\left(\gamma^0 p_0 + \gamma^1 p_1 + \gamma^2 p_2 + \gamma^3 p_3 - \hat{1}Mc + \gamma^0 \vec{p} \frac{\dot{\vec{r}}}{c} \right) \varphi_t(\vec{p}, t) = 0. \quad (233)$$

With the velocity as a function of the momentum

$$\vec{p} = \frac{M\dot{\vec{r}}}{\sqrt{1 - \frac{\dot{\vec{r}}^2}{c^2}}}, \quad \frac{\dot{\vec{r}}}{c} = \frac{\vec{p}}{\sqrt{M^2 c^2 + \vec{p}^2}}, \quad (234)$$

for the last terms of these equations we obtain an expression depending only on momentum,

$$\vec{p} \frac{\dot{\vec{r}}}{c} = \frac{\vec{p}^2}{\sqrt{M^2 c^2 + \vec{p}^2}} = \frac{\vec{p}^2}{E} \quad (235)$$

With the standard notations

$$\not{\partial} = \gamma^0 \frac{\partial}{\partial x^0} + \gamma^1 \frac{\partial}{\partial x^1} + \gamma^2 \frac{\partial}{\partial x^2} + \gamma^3 \frac{\partial}{\partial x^3} = \gamma^\mu \partial_\mu$$

$$\not{p} = \gamma^0 p_0 + \gamma^1 p_1 + \gamma^2 p_2 + \gamma^3 p_3 = \gamma^\mu p_\mu, \quad (236)$$

and $m = Mc$, the dynamic equations (233) take the form

$$\left[-i\hbar \not{\partial} - m \left(\hat{1} - \gamma^0 \eta \right) \right] \psi_t(x) = 0$$

$$\left[\not{p} - m \left(\hat{1} - \gamma^0 \eta \right) \right] \varphi_t(p) = 0 \quad (237)$$

which, compared to the similar equations obtained from the Schrödinger-Dirac equation^{5,6}, include an additional term depending on momentum,

$$\eta = \vec{p} \frac{\dot{\vec{r}}}{mc} = \frac{\vec{p}^2}{mE} \quad (238)$$

For a four-dimensional wave function with the two-dimensional components $\tilde{u}(p)$ and $\tilde{v}(p)$,

$$\varphi_t(p) \sim u(p) = \begin{pmatrix} u_1(p) \\ \tilde{u}(p) \\ \tilde{v}(p) \\ u_4(p) \end{pmatrix}, \quad (239)$$

the second dynamic equation (233) in the momentum space is

$$\left[\gamma^0 p_0 + \gamma^1 p_1 + \gamma^2 p_2 + \gamma^3 p_3 - m(\hat{1} - \gamma^0 \eta) \right] \begin{pmatrix} u_1(p) \\ u_2(p) \\ u_3(p) \\ u_4(p) \end{pmatrix} = 0. \quad (240)$$

With the Clifford algebra elements (226),

$$\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, \gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad (241)$$

for a reference system with the z -axis in the direction of momentum p_3 of the particle⁶, we obtain the dynamic matrix equation

$$\begin{pmatrix} p_0 + m\eta - m & 0 & p_3 & 0 \\ 0 & p_0 + m\eta - m & 0 & -p_3 \\ -p_3 & 0 & -(p_0 + m\eta + m) & 0 \\ 0 & p_3 & 0 & -(p_0 + m\eta + m) \end{pmatrix} \begin{pmatrix} u_1(p) \\ u_2(p) \\ u_3(p) \\ u_4(p) \end{pmatrix} = 0. \quad (242)$$

From the condition of a null determinant of this system of equations for a nontrivial solution,

$$\begin{vmatrix} p_0 - m(1-\eta) & 0 & p_3 & 0 \\ 0 & p_0 - m(1-\eta) & 0 & -p_3 \\ -p_3 & 0 & -[p_0 + m(1+\eta)] & 0 \\ 0 & p_3 & 0 & -[p_0 + m(1+\eta)] \end{vmatrix} \\ = [p_0 - m(1-\eta)] \{ [p_0 - m(1-\eta)] [p_0 + m(1+\eta)]^2 - p_3^2 [p_0 + m(1+\eta)] \} \\ + p_3 [p_3^3 - [p_0 + m(1+\eta)] [p_0 - m(1-\eta)] p_3] \\ = [(p_0 + m\eta)^2 - p_3^2 - m^2]^2 = 0, \quad (243)$$

we obtain the relation

$$p_0 + m\eta = \pm E, \quad (244)$$

where

$$E = \sqrt{p_3^2 + m^2}, \quad (245)$$

for a particle with the energy cE . With a positive energy (244), $E_+ = E$, for the two-dimensional components of the wave function (239), from (242) with the expressions of the Pauli spin matrices we obtain the dynamic equations

$$\begin{aligned} (E_+ - m)\tilde{u}_+(p) + \sigma_3 p_3 \tilde{v}_+(p) &= 0 \\ \sigma_3 p_3 \tilde{u}_+(p) + (E_+ + m)\tilde{v}_+(p) &= 0. \end{aligned} \quad (246)$$

We consider a normalized spin wave function $\tilde{u}_+(p)$,
 $\tilde{u}_+^\dagger(p)\tilde{u}_+(p) = 1$,

$$(247)$$

as from the second equation (246) we obtain the spin-momentum wave function

$$\tilde{v}_+(p) = -\frac{\sigma_3 p_3}{E_+ + m} \tilde{u}_+(p) = \frac{\sigma_3 p^3}{E + m} \tilde{u}_+(p). \quad (248)$$

For a negative energy (244) of an antiparticle, $E_- = -E$, from (242) we obtain the dynamic equations

$$\begin{aligned} (E_- - m)\tilde{u}_-(p) + \sigma_3 p_3 \tilde{v}_-(p) &= 0 \\ \sigma_3 p_3 \tilde{u}_-(p) + (E_- + m)\tilde{v}_-(p) &= 0, \end{aligned} \quad (249)$$

where we consider a normalized spin wave function $\tilde{v}_-(p)$,
 $\tilde{v}_-^\dagger(p)\tilde{v}_-(p) = 1$,

$$(250)$$

as from the first equation (249) we obtain

$$\tilde{u}_-(p) = -\frac{\sigma_3 p_3}{E_- - m} \tilde{v}_-(p) = -\frac{\sigma_3 p^3}{E + m} \tilde{v}_-(p). \quad (251)$$

We normalize the wave function (239) with the particle and antiparticle components (248) and (251) in an arbitrary system of reference, with the coefficients α and β ,

$$u(p) = \alpha u_+(p) + \beta u_-(p), \quad u_+(p) = \begin{pmatrix} \tilde{u}_+(p) \\ \frac{\vec{\sigma} \vec{p}}{E + m} \tilde{u}_+(p) \end{pmatrix}, \quad u_-(p) = \begin{pmatrix} -\frac{\vec{\sigma} \vec{p}}{E + m} \tilde{v}_-(p) \\ \tilde{v}_-(p) \end{pmatrix}. \quad (252)$$

From the amplitude squares of these wave components with the Hermiticity of the matrix $\vec{\sigma} \vec{p}$ and the null anticommutators of the Pauli spin matrices, we obtain the expression

$$\frac{\vec{p}\vec{\sigma}}{E+m} \frac{\vec{\sigma}\vec{p}}{E+m} = \frac{\vec{p}^2}{(E+m)^2} = \frac{E-m}{E+m}, \quad (253)$$

which leads to the normalization conditions of the two components,

$$\begin{aligned} u_+^\dagger(p)u_+(p) &= \begin{pmatrix} \tilde{u}_+^\dagger(p) & \tilde{u}_+^\dagger(p)\frac{\vec{\sigma}\vec{p}}{E+m} \end{pmatrix} \begin{pmatrix} \tilde{u}_+(p) \\ \frac{\vec{\sigma}\vec{p}}{E+m}\tilde{u}_+(p) \end{pmatrix} = \left(1 + \frac{E-m}{E+m}\right) \tilde{u}_+^\dagger(p)\tilde{u}_+(p) \\ &= \frac{2E}{E+m} \end{aligned} \quad (254)$$

and

$$\begin{aligned} u_-^\dagger(p)u_-(p) &= \begin{pmatrix} -\tilde{v}_-^\dagger(p)\frac{\vec{\sigma}\vec{p}}{E+m} & \tilde{v}_-^\dagger(p) \end{pmatrix} \begin{pmatrix} -\frac{\vec{\sigma}\vec{p}}{E+m}\tilde{v}_-(p) \\ \tilde{v}_-(p) \end{pmatrix} = \left(\frac{E-m}{E+m} + 1\right) \tilde{v}_-^\dagger(p)\tilde{v}_-(p) \\ &= \frac{2E}{E+m}. \end{aligned} \quad (255)$$

With these conditions, and the condition of a covariant normalization⁶ of the total spinor (252),

$$u^\dagger(p)u(p) = \left[|\alpha|^2 \tilde{u}_+^\dagger(p)\tilde{u}_+(p) + |\beta|^2 \tilde{v}_-^\dagger(p)\tilde{v}_-(p) \right] \frac{2E}{E+m} = \frac{E}{m}, \quad (256)$$

for an arbitrary momentum $\vec{p}$ and the normalization conditions (247) and (250) of the particle and antiparticle spin wave functions, we obtain the normalization condition

$$|\alpha|^2 + |\beta|^2 = \frac{E+m}{2m}. \quad (257)$$

With relation (239), we obtain the momentum wave function (221) in a momentum one wave approximation over the spectral domain $\Delta^3\vec{p}$ of the momentum space, with a normalization width L_p ,

$$\varphi(\vec{p}, t) = e^{\frac{i}{\hbar}\vec{p}\vec{r}} \varphi_t(p) = \frac{1}{L_p^{3/2}} e^{\frac{i}{\hbar}\vec{p}\vec{r}} u(p) = \varphi(\vec{p}), \quad \vec{p} \in \Delta^3\vec{p}. \quad (258)$$

With this wave function, the wave function (221) of the quantum particle in the coordinate space is

$$\psi(\vec{r}, t) = \frac{1}{(2\pi\hbar L_p)^{3/2}} \int_{\Delta^3\vec{p}} u(p) e^{\frac{i}{\hbar}[\vec{p}\vec{r} - H(\vec{p}, \vec{r})]t} d^3\vec{p}. \quad (259)$$

For a free particle with coordinates $\vec{r} = \dot{\vec{r}}t$, we obtain a coordinate wave function in the momentum one wave approximation (258), with the particle and antiparticle components (252),

$$\psi(\vec{r}, t) = \frac{1}{(2\pi\hbar L_p)^{3/2}} \int_{\Delta^3 \vec{p}} e^{-\frac{i}{\hbar} \vec{p} \cdot \vec{r}} \left[\alpha u_+(p) e^{\frac{i}{\hbar} cEt} + \beta u_-(p) e^{-\frac{i}{\hbar} cEt} \right] d^3 \vec{p} \quad (260)$$

with the normalization condition which, with (254), (255) and (257), is

$$L_p^3 = \int_{\Delta^3 \vec{p}} \frac{2E}{E+m} (|\alpha|^2 + |\beta|^2) d^3 \vec{p} = \int_{\Delta^3 \vec{p}} \frac{2E}{E+m} \frac{E+m}{2m} d^3 \vec{p} = \frac{1}{m} \int_{\Delta^3 \vec{p}} E d^3 \vec{p} \quad (261)$$

By an extension of the momentum spectrum for the components $u_+(p)$, $u_-(p)$ of this wave function, the plane waves (223) infinitely extended in space, are reduced to wave-functions concentrated in space, specific to a particle. Since the two momentum eigenfunctions $u_+(p)$ and $u_-(p)$ do not depend on time, and the width of the momentum spectrum can be considered much smaller than the momentum value, $\Delta \vec{p} \ll |\vec{p}|$, the wave packet (260) describes an invariant distribution of matter, with a spatial width $\Delta \vec{r}$, mainly determined by the propagation factor with the phase $\vec{r} \cdot \vec{p}$, propagating with the group velocity

$$\frac{\partial}{\partial \vec{p}} cE = \frac{\partial}{\partial \vec{p}} c \sqrt{M^2 c^2 + \vec{p}^2} = \frac{c \vec{p}}{\sqrt{M^2 c^2 + \vec{p}^2}} = \dot{\vec{r}} \quad (262)$$

We notice that the amplitude of the momentum component of the state four-vector (252) tends to 1,

$$\tilde{u}_+^\dagger(p) \frac{\vec{\sigma} \cdot \vec{p}}{E+m} \frac{\vec{\sigma} \cdot \vec{p}}{E+m} \tilde{u}_+(p) = \frac{E-m}{E+m} \tilde{u}_+^\dagger(p) \tilde{u}_+(p) \xrightarrow{|\vec{p}| \gg m} \tilde{u}_+^\dagger(p) \tilde{u}_+(p) = 1 \quad (263)$$

which means that for a velocity increase from null to the light velocity the momentum function amplitude increases from null to amplitude 1 of the spin component. Finally, we notice that with the relation

$$\psi_t(x) \sim e^{\frac{i}{\hbar} px} \varphi_t(p) = \frac{1}{L_p^{3/2}} e^{\frac{i}{\hbar} px} u(p) \quad (264)$$

depending on the momentum-coordinate scalar product

$$px = p_0 x^0 + p_1 x^1 + p_2 x^2 + p_3 x^3 \quad (265)$$

the two equations (237), obtained for the two wave functions (221) in the two conjugated spaces of the coordinates and momentum, take the same form,

$$-i\hbar \not{\partial} e^{\frac{i}{\hbar} px} = (\gamma^0 p_0 + \gamma^1 p_1 + \gamma^2 p_2 + \gamma^3 p_3) e^{\frac{i}{\hbar} px} = \not{p} e^{\frac{i}{\hbar} px} \quad (266)$$

CONCLUSIONS

We noticed that the conventional Schrödinger equation describes only a time dependent factor of the total particle dynamics, in contradiction with the fundamental Hamiltonian dynamics, and without taking into account the propagation in a free space, which, by a finite momentum spectrum, gives spatial dimensions to this particle. The solution of the Schrödinger equation of

a free particle, rapidly spreading to infinite dimensions, leads to the hardly acceptable interpretation of the matter as a system of punctual particles, mysteriously appearing and disappearing in space, with this solution as a probability amplitude. When a propagation factor is considered in the coordinate space for a wave function, as a packet of the momentum eigenstates, we also have to consider the inverse Fourier transform, as another wave packet, in the conjugated space of this momentum. The group velocities of these wave packets take the form of the Hamilton equations only when the Hamiltonian in the time dependent phase, obtained from the Schrödinger equation, is replaced by the Lagrangian function. It is remarkable that when the relativistic Lagrangian is considered in the proper time of a free particle, this Lagrangian is a constant $L = -Mc^2$. This means that the spatial distribution of this wave function does not depend on time, but only on the product momentum-coordinates in the coordinate dependent phase. In this way, a free particle appears as an invariant distribution of matter propagating in space according to the general theory of relativity. Under the action of an external field, from the Lagrangian we obtain a velocity perpendicular to this field. Considering the mass M , which describes the dynamics of the matter density, equal to the mass M_0 as the integral of this density, we obtained the quantization condition, of a particle with the total mass given by the dynamics of its density. For the realistic case of a particle much smaller than the dimensions of any curvature variations of the time-space system, we obtained wave functions in the time of a local system. Thus, a system of particles is in fact a continuous distribution of matter, with dynamics described by quanta with the mass given by the dynamic properties of the matter density, half integer spins for Fermions, and integer spins for Bosons. From the mass invariance for a coordinate transformation, we obtained an equation of conservation for the particle matter in an arbitrary system of reference.

We described the interaction of a quantum particle with an electromagnetic field by additional terms in the relativistic time-dependent phases of its wave functions, with a vector potential conjugated to coordinates, as for the Aharonov-Bohm effect, and a scalar potential conjugated to time. We obtained the Lorentz force for the electric and magnetic fields as functions of these potentials. From these expressions, we obtained three of the four Maxwell equations: the two Gauss-Maxwell laws for the electric and magnetic fluxes, and the Faraday-Maxwell law of electromagnetic induction. The fourth Maxwell law, Ampère-Maxwell, is obtained from the condition of a light velocity equal to the maximum velocity of a relativistic particle. From this equation as a function of the electric charge density, we obtain the charge conservation.

From our new Schrödinger-type equation in the coordinate space with the relativistic Dirac's Hamiltonian, we obtained relativistic equations that, compared to Dirac's equations, include explicit terms depending on velocity, as should be in relativistic theory. For a nonrelativistic quantum particle in an electric field with a scalar potential $U(x^i)$, interacting with electromagnetic radiation with vector potential $A^k(x^i, t)$, and a gravitational wave with the gravitational potential of the metric tensor $g^{\mu\nu}$, we obtained a wave packet of rapidly varying waves, with a large frequency determined by the rest energy, modulated by a frequency depending on these potentials, and a wave vector depending on the vector potential and the particle velocity. With the group velocity, we found a wave vector proportional to the vector

potential of the electromagnetic radiation, and a modulation frequency mainly depending on the electric potential and the potential of the gravitational wave.

We integrated the geodesic equations for a quantum particle in a central gravitational field. For a particle at a distance r larger than the Schwarzschild radius r_0 , we obtained a velocity directed to the gravitational centre, but decelerated for $r_0 < r < 3r_0$, without reaching the Schwarzschild boundary. For a particle in a black hole, $r < r_0 = 2m$, the velocity is directed towards the Schwarzschild boundary, but reaches this boundary only in a time tending to infinity. When, by the strong gravitational field of a large agglomeration of matter, this matter concentrates inside a radius r_G smaller than the Schwarzschild radius r_0 , all the matter structures, such as macroscopic bodies, atoms, and nuclei, are crushed up in this field, to a uniform distribution of matter, favouring spherical symmetry. When such a structure arises, the matter at the centre of this structure explodes with a velocity much larger than the light velocity, but reaches this velocity somewhere not very far from half of its radius. In this framework, our Universe seems to be a huge black hole, starting with an explosion process called Big Bang-inflation. Unlike the standard Big Bang-inflation process, with the whole matter of the Universe initially concentrated as a single point, our Big Bang-inflation process is an explosion, with a velocity much larger than the light velocity, of the matter in the central region of a huge agglomeration of matter, which, by crushing up the objects in its very strong gravitational field, spontaneously reached spherical symmetry. The velocity field in the gravitational potential of a black hole, tends to concentrate at the Schwarzschild boundary the matter from the inside, tending to empty the inner part of this black hole, and from the outside, continuously enlarging its radius r_0 . However, due to violent collisions of the bodies starting from the central regions, with velocities much higher than the velocities of the bodies starting from more distant places, after the Big Bang the cosmic systems were scattered throughout the whole Universe, as quasi-inertial systems due to their huge radius r_0 . The light emitted near the Schwarzschild boundary is redshifted by the increase in the gravitational field with the distance from the centre of this black hole.

We considered a quantum particle in a gravitational wave, for the realistic, nonrelativistic, case, and found that the particle acceleration is proportional to the gradient of the metric tensor element g_{00} . In the first-order approximation of the metric tensor as a solution of the gravitational wave equation, this acceleration is a constant in the direction of propagation of the gravitational wave. From the normalization condition of the polarization vector, for the amplitude elements of a metric tensor in a gravitational wave propagating in a direction x^3 , we found a rotation invariant of the amplitude components in the perpendicular plane (x^1, x^2) , with an angular momentum $S_t = 2$, which we call the graviton spin. We noticed that the time-space interval invariant, as the time-dependent phase of the matter density amplitude function we call the wave function, is the product of two rotation invariants: a gravitational invariant with an angular momentum called the graviton spin, and a coordinate invariant, with an angular momentum called the particle spin, with a half-integer value for Fermions and an integer value

for Bosons – the spin-statistics relation. In the second-order approximation of the metric tensor, as a solution of the gravitational wave equation, for a quantum particle we found an oscillation in the oscillation direction of this wave.

From our dynamic equations in the two conjugated spaces of the coordinates and momentum, we reobtained the two dynamical equations of the quantum field theory for a free particle, but with a change of sign for the derivatives with the coordinates, and an additional term, depending on the momentum, for the eigenvalues of these equations. However, when these eigenvalues are eliminated, we reobtain the spinor expressions of the conventional quantum field theory. In this framework, we consider finite quantum particles, as a wave packet of the momentum eigenfunctions for particles and antiparticles, with a normalization coefficient given by the ratio of the total energy integral to the rest energy, over the domain of the momentum spectrum, which gives the matter distribution of the particle.

Essentially, in this paper we described a quantum particle by wave packets in the coordinate and momentum spaces, as amplitudes of the matter density, with group velocities describing their dynamics, and rotation invariants of the time dependent phases, for the graviton spin and the particle matter spin. In a local system, the particle time dependent phase is an invariant with a time dependent term larger than the coordinate dependent term, describing a motion with a maximum velocity, called light velocity. In the system of a black hole, the radius dependent term becomes larger than the time dependent term, and the velocity of a quantum particle inside this black hole takes values much larger than the light velocity. We reobtained the Big Bang-inflation process, the dark matter of the particles without any visible charge, the dark energy, as the energy of the matter travelling towards the Schwarzschild boundary in the inflation process after Big Bang, the universal expansion by the matter accumulation at the Schwarzschild boundary from the outside, and the light redshift, as a gravitational effect by motion towards this boundary. We also obtained the free particle states as wave packets of spinors according to quantum field theory.

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