

Periodic Oscillation for an Eight-Neuron BAM Neural Network Model with Discrete Delays

Chunhua Feng

Department of Mathematics and Computer Science,
Alabama State University, Montgomery, AL, USA, 36104

ABSTRACT

This paper investigates the existence of periodic oscillations for an eight-neuron BAM neural network model with discrete delays. Two theorems are provided to guarantee the existence of periodic oscillations for this model by using mathematical analysis method, which is simpler than bifurcation method. The criteria for selecting of the parameters in this network are provided. Computer simulation examples are presented to demonstrate the correctness of this method.

Keywords: eight-neuron BAM network model, delay, instability, periodic solution

INTRODUCTION

Due to various applications of neural networks, the dynamics behaviors of neural networks with delays have attracted great attention of many researchers [1-21]. In particular, there are extensive literatures on simplified bidirectional associative memory (BAM) neural networks. Various interesting results on periodic solution have been reported. For example, in 2006, Yu and Cao have considered the Hopf bifurcation for the following four-neuron BAM neural network model with discrete delays [1]:

$$\begin{cases} x_1'(t) = -\mu_1 x_1(t) + c_{11} f_{11}(y_1(t - \tau_3)) + c_{12} f_{12}(y_2(t - \tau_3)), \\ x_2'(t) = -\mu_2 x_2(t) + c_{21} f_{21}(y_1(t - \tau_4)) + c_{22} f_{22}(y_2(t - \tau_4)), \\ y_1'(t) = -\mu_3 y_1(t) + d_{11} g_{11}(x_1(t - \tau_1)) + d_{12} g_{12}(x_2(t - \tau_2)), \\ y_2'(t) = -\mu_4 y_2(t) + d_{21} g_{21}(x_1(t - \tau_1)) + d_{22} g_{22}(x_2(t - \tau_2)). \end{cases} \quad (1)$$

The existence of Hopf bifurcation, a formula for determining of the Hopf bifurcation and the stability of bifurcating periodic solution have been obtained. Ge and Xu have investigated a five-neuron BAM network model with delays as follows [2]:

$$\begin{cases} x_1'(t) = -\mu x_1(t) + c_{11} f(y_1(t - \tau_3)) + c_{12} f(y_2(t - \tau_3)) + c_{13} f(y_3(t - \tau_3)), \\ x_2'(t) = -\mu x_2(t) + c_{21} f(y_1(t - \tau_4)) + c_{22} f(y_2(t - \tau_4)) + c_{23} f(y_3(t - \tau_4)), \\ y_1'(t) = -\mu y_1(t) + d_{11} f(x_1(t - \tau_1)) + d_{12} f(x_2(t - \tau_2)), \\ y_2'(t) = -\mu y_2(t) + d_{21} f(x_1(t - \tau_1)) + d_{22} f(x_2(t - \tau_2)), \\ y_3'(t) = -\mu y_3(t) + d_{31} f(x_1(t - \tau_1)) + d_{32} f(x_2(t - \tau_2)). \end{cases} \quad (2)$$

Some sufficient conditions for the synchronization and bifurcation were exhibited. The global attractivity of the trivial solution were also established. Xu et al. extended model (1) to six dimensional delayed BAM network system [3]:

$$\begin{cases} x_1'(t) = -\mu_1 x_1(t) + c_{11}f_{11}(y_1(t - \tau_4)) + c_{12}f_{12}(y_2(t - \tau_4)) + c_{13}f_{13}(y_3(t - \tau_4)), \\ x_2'(t) = -\mu_2 x_2(t) + c_{21}f_{21}(y_1(t - \tau_5)) + c_{22}f_{22}(y_2(t - \tau_5)) + c_{23}f_{23}(y_3(t - \tau_5)), \\ x_3'(t) = -\mu_3 x_3(t) + c_{31}f_{31}(y_1(t - \tau_6)) + c_{32}f_{32}(y_2(t - \tau_6)) + c_{33}f_{33}(y_3(t - \tau_6)), \\ y_1'(t) = -\mu_4 y_1(t) + c_{41}f_{41}(x_1(t - \tau_1)) + c_{42}f_{42}(x_2(t - \tau_2)) + c_{43}f_{43}(x_3(t - \tau_3)), \\ y_2'(t) = -\mu_5 y_2(t) + c_{51}f_{51}(x_1(t - \tau_1)) + c_{52}f_{52}(x_2(t - \tau_2)) + c_{53}f_{53}(x_3(t - \tau_3)), \\ y_3'(t) = -\mu_6 y_3(t) + c_{61}f_{61}(x_1(t - \tau_1)) + c_{62}f_{62}(x_2(t - \tau_2)) + c_{63}f_{63}(x_3(t - \tau_3)). \end{cases} \quad (3)$$

By analyzing the associated characteristic transcendental equation, the linear stability of the model and Hopf bifurcation were demonstrated by using the normal form method and center manifold theory. However, it was pointed out that by using the bifurcating method to discuss the existence of bifurcating periodic solution it is necessary to make some restrictive conditions such that the neural network contains only one delay. For example, in model (3) the authors assumed that,

$\tau_1 + \tau_4 = \tau_2 + \tau_5 = \tau_3 + \tau_6 = \tau$, and let $u_1(t) = x_1(t - \tau_1)$, $u_2(t) = x_2(t - \tau_2)$, $u_3(t) = x_3(t - \tau_3)$, $u_4(t) = y_1(t)$, $u_5(t) = y_2(t)$, $u_6(t) =$

$y_3(t)$, and then model (3) changes to only one delay system as the follows:

$$\begin{cases} u_1'(t) = -\mu_1 u_1(t) + c_{11}f_{11}(u_4(t - \tau)) + c_{12}f_{12}(u_5(t - \tau)) + c_{13}f_{13}(u_6(t - \tau)), \\ u_2'(t) = -\mu_2 u_2(t) + c_{21}f_{21}(u_4(t - \tau)) + c_{22}f_{22}(u_5(t - \tau)) + c_{23}f_{23}(u_6(t - \tau)), \\ u_3'(t) = -\mu_3 u_3(t) + c_{31}f_{31}(u_4(t - \tau)) + c_{32}f_{32}(u_5(t - \tau)) + c_{33}f_{33}(u_6(t - \tau)), \\ u_4'(t) = -\mu_4 u_4(t) + c_{41}f_{41}(u_1(t)) + c_{42}f_{42}(u_2(t)) + c_{43}f_{43}(u_3(t)), \\ u_5'(t) = -\mu_5 u_5(t) + c_{51}f_{51}(u_1(t)) + c_{52}f_{52}(u_2(t)) + c_{53}f_{53}(u_3(t)), \\ u_6'(t) = -\mu_6 u_6(t) + c_{61}f_{61}(u_1(t)) + c_{62}f_{62}(u_2(t)) + c_{63}f_{63}(u_3(t)). \end{cases} \quad (4)$$

Thus, the bifurcating equation can be discussed. In this paper, we extend model (3) to an eight-neuron BAM system:

$$\begin{cases} x_1'(t) = -\mu_1 x_1(t) + c_{11}f_{11}(y_1(t - \tau_5)) + c_{12}f_{12}(y_2(t - \tau_6)) + c_{13}f_{13}(y_3(t - \tau_7)) + c_{14}f_{14}(y_4(t - \tau_8)), \\ x_2'(t) = -\mu_2 x_2(t) + c_{21}f_{21}(y_1(t - \tau_5)) + c_{22}f_{22}(y_2(t - \tau_6)) + c_{23}f_{23}(y_3(t - \tau_7)) + c_{24}f_{24}(y_4(t - \tau_8)), \\ x_3'(t) = -\mu_3 x_3(t) + c_{31}f_{31}(y_1(t - \tau_5)) + c_{32}f_{32}(y_2(t - \tau_6)) + c_{33}f_{33}(y_3(t - \tau_7)) + c_{34}f_{34}(y_4(t - \tau_8)), \\ x_4'(t) = -\mu_4 x_4(t) + c_{41}f_{41}(y_1(t - \tau_5)) + c_{42}f_{42}(y_2(t - \tau_6)) + c_{43}f_{43}(y_3(t - \tau_7)) + c_{44}f_{44}(y_4(t - \tau_8)), \\ y_1'(t) = -\mu_5 y_1(t) + c_{51}f_{51}(x_1(t - \tau_1)) + c_{52}f_{52}(x_2(t - \tau_2)) + c_{53}f_{53}(x_3(t - \tau_3)) + c_{54}f_{54}(x_4(t - \tau_4)), \\ y_2'(t) = -\mu_6 y_2(t) + c_{61}f_{61}(x_1(t - \tau_1)) + c_{62}f_{62}(x_2(t - \tau_2)) + c_{63}f_{63}(x_3(t - \tau_3)) + c_{64}f_{64}(x_4(t - \tau_4)), \\ y_3'(t) = -\mu_7 y_3(t) + c_{71}f_{71}(x_1(t - \tau_1)) + c_{72}f_{72}(x_2(t - \tau_2)) + c_{73}f_{73}(x_3(t - \tau_3)) + c_{74}f_{74}(x_4(t - \tau_4)), \\ y_4'(t) = -\mu_8 y_4(t) + c_{81}f_{81}(x_1(t - \tau_1)) + c_{82}f_{82}(x_2(t - \tau_2)) + c_{83}f_{83}(x_3(t - \tau_3)) + c_{84}f_{84}(x_4(t - \tau_4)). \end{cases} \quad (5)$$

where $-\mu_i < 0$ ($i = 1, 2, \dots, 8$). Our goal is to consider the dynamical behavior of model (5). Noting that system (5) has eight delays. If those delays are different positive numbers, then the bifurcating method is extremely hard to deal with model (5). Because one cannot solve the bifurcating equation which is an eight variables transcendental equation. In order to discuss the existence of periodic solutions for system (5), we adopt the generalized Chafee's criterion [23], and the appendix of [24]. and this particular instability of the unique equilibrium point and the boundedness of the solutions will force system (5) to generate a limit cycle, namely, a periodic solution. In this paper, by using the mathematical analysis method, the existence of periodic solutions has been established.

Preliminaries

For the activation functions f_{ij} , we assume that f_{ij} are continuous bounded differentiable functions, satisfying:

$$f_{ij}(0) = 0, \quad u f_{ij}(u) > 0 \quad (u \neq 0) \quad (6)$$

The general activation functions such as $\tanh(x)$, $\arctan(x)$ satisfy condition (6). The linearized system of (5) is the following:

$$\begin{cases} x'_1(t) = -\mu_1 x_1(t) + b_{11} y_1(t - \tau_5) + b_{12} y_2(t - \tau_6) + b_{13} y_3(t - \tau_7) + b_{14} y_4(t - \tau_8), \\ x'_2(t) = -\mu_2 x_2(t) + b_{21} y_1(t - \tau_5) + b_{22} y_2(t - \tau_6) + b_{23} y_3(t - \tau_7) + b_{24} y_4(t - \tau_8), \\ x'_3(t) = -\mu_3 x_3(t) + b_{31} y_1(t - \tau_5) + b_{32} y_2(t - \tau_6) + b_{33} y_3(t - \tau_7) + b_{34} y_4(t - \tau_8), \\ x'_4(t) = -\mu_4 x_4(t) + b_{41} y_1(t - \tau_5) + b_{42} y_2(t - \tau_6) + b_{43} y_3(t - \tau_7) + b_{44} y_4(t - \tau_8), \\ y'_1(t) = -\mu_5 y_1(t) + b_{51} x_1(t - \tau_1) + b_{52} x_2(t - \tau_2) + b_{53} x_3(t - \tau_3) + b_{54} x_4(t - \tau_4), \\ y'_2(t) = -\mu_6 y_2(t) + b_{61} x_1(t - \tau_1) + b_{62} x_2(t - \tau_2) + b_{63} x_3(t - \tau_3) + b_{64} x_4(t - \tau_4), \\ y'_3(t) = -\mu_7 y_3(t) + b_{71} x_1(t - \tau_1) + b_{72} x_2(t - \tau_2) + b_{73} x_3(t - \tau_3) + b_{74} x_4(t - \tau_4), \\ y'_4(t) = -\mu_8 y_4(t) + b_{81} x_1(t - \tau_1) + b_{82} x_2(t - \tau_2) + b_{83} x_3(t - \tau_3) + b_{84} x_4(t - \tau_4). \end{cases} \quad (7)$$

where $b_{ij} = c_{ij} f'_{ij}(0)$, $i, j = 1, 2, \dots, 8$. For convenience, let $u_i(t) = x_i(t)$ ($i = 1, \dots, 4$) and $u_5(t) = y_1(t)$, $u_6(t) = y_2(t)$, $u_7(t) = y_3(t)$, $u_8(t) = y_4(t)$, then system (7) can be rewritten as the follows:

$$\begin{cases} u'_1(t) = -\mu_1 u_1(t) + b_{11} u_5(t - \tau_5) + b_{12} u_6(t - \tau_6) + b_{13} u_7(t - \tau_7) + b_{14} u_8(t - \tau_8), \\ u'_2(t) = -\mu_2 u_2(t) + b_{21} u_5(t - \tau_5) + b_{22} u_6(t - \tau_6) + b_{23} u_7(t - \tau_7) + b_{24} u_8(t - \tau_8), \\ u'_3(t) = -\mu_3 u_3(t) + b_{31} u_5(t - \tau_5) + b_{32} u_6(t - \tau_6) + b_{33} u_7(t - \tau_7) + b_{34} u_8(t - \tau_8), \\ u'_4(t) = -\mu_4 u_4(t) + b_{41} u_5(t - \tau_5) + b_{42} u_6(t - \tau_6) + b_{43} u_7(t - \tau_7) + b_{44} u_8(t - \tau_8), \\ u'_5(t) = -\mu_5 u_5(t) + b_{51} u_1(t - \tau_1) + b_{52} u_2(t - \tau_2) + b_{53} u_3(t - \tau_3) + b_{54} u_4(t - \tau_4), \\ u'_6(t) = -\mu_6 u_6(t) + b_{61} u_1(t - \tau_1) + b_{62} u_2(t - \tau_2) + b_{63} u_3(t - \tau_3) + b_{64} u_4(t - \tau_4), \\ u'_7(t) = -\mu_7 u_7(t) + b_{71} u_1(t - \tau_1) + b_{72} u_2(t - \tau_2) + b_{73} u_3(t - \tau_3) + b_{74} u_4(t - \tau_4), \\ u'_8(t) = -\mu_8 u_8(t) + b_{81} u_1(t - \tau_1) + b_{82} u_2(t - \tau_2) + b_{83} u_3(t - \tau_3) + b_{84} u_4(t - \tau_4). \end{cases} \quad (8)$$

The linearized system (8) can be written in a matrix form:

$$U'(t) = AU(t) + B U(t - \tau) \quad (9)$$

where $U(t) = [u_1(t), u_2(t), \dots, u_8(t)]^T$, $U(t - \tau) = [u_1(t - \tau_1), u_2(t - \tau_2), \dots, u_8(t - \tau_8)]^T$. $A = \text{diag}(-\mu_1, -\mu_2, \dots, -\mu_8)$, and B is an eight by eight matrix

$$B = (b_{ij})_{8 \times 8} = \begin{bmatrix} 0 & 0 & 0 & 0 & b_{11} & b_{12} & b_{13} & b_{14} \\ 0 & 0 & 0 & 0 & b_{21} & b_{22} & b_{23} & b_{24} \\ 0 & 0 & 0 & 0 & b_{31} & b_{32} & b_{33} & b_{34} \\ 0 & 0 & 0 & 0 & b_{41} & b_{42} & b_{43} & b_{44} \\ b_{51} & b_{52} & b_{53} & b_{54} & 0 & 0 & 0 & 0 \\ b_{61} & b_{62} & b_{63} & b_{64} & 0 & 0 & 0 & 0 \\ b_{71} & b_{72} & b_{73} & b_{74} & 0 & 0 & 0 & 0 \\ b_{81} & b_{82} & b_{83} & b_{84} & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Lemma 1 All solutions of system (5) are bounded.

Proof From condition (6), the activation functions are continuous bounded, and assuming that $|f_{ij}(u)| \leq N_{ij}$ ($i = 1, 2, \dots, 8$), then

$$\frac{d|u_i(t)|}{dt} \leq -\mu_i |u_i(t)| + N_i, \quad N_i = \sum_{j=1}^4 |b_{ij}| N_{ij} \quad (i = 1, 2, \dots, 8) \quad (10)$$

From (10), noting that μ_i ($i = 1, 2, \dots, 8$) are positive real numbers, we obtain

$$|u_i(t)| \leq \frac{N_i}{\mu_i} \quad (i = 1, 2, \dots, 8) \quad (11)$$

This system (8) has a unique equilibrium point means that the solutions of system (5) are uniformly bounded.

Lemma 2 If the matrix B is not a positive definite matrix, then system (5) has a unique equilibrium point.

Proof First we show that system (8) has a unique equilibrium point. An equilibrium point $U^* = [u_1^*, u_2^*, \dots, u_8^*]^T$ is a solution of the following algebraic equation:

$$\begin{cases} -\mu_1 u_1^* + b_{11} u_5^* + b_{12} u_6^* + b_{13} u_7^* + b_{14} u_8^* = 0, \\ -\mu_2 u_2^* + b_{21} u_5^* + b_{22} u_6^* + b_{23} u_7^* + b_{24} u_8^* = 0, \\ -\mu_3 u_3^* + b_{31} u_5^* + b_{32} u_6^* + b_{33} u_7^* + b_{34} u_8^* = 0, \\ -\mu_4 u_4^* + b_{41} u_5^* + b_{42} u_6^* + b_{43} u_7^* + b_{44} u_8^* = 0, \\ -\mu_5 u_5^* + b_{51} u_1^* + b_{52} u_2^* + b_{53} u_3^* + b_{54} u_4^* = 0, \\ -\mu_6 u_6^* + b_{61} u_1^* + b_{62} u_2^* + b_{63} u_3^* + b_{64} u_4^* = 0, \\ -\mu_7 u_7^* + b_{71} u_1^* + b_{72} u_2^* + b_{73} u_3^* + b_{74} u_4^* = 0, \\ -\mu_8 u_8^* + b_{81} u_1^* + b_{82} u_2^* + b_{83} u_3^* + b_{84} u_4^* = 0. \end{cases} \quad (12)$$

System (12) can be written as a matrix form

$$BU^* = -AU^* \quad (13)$$

Obviously, there is a trivial solution of system (12), implying that zero is an equilibrium point of system (8). Noting that $-A$ is a positive definite matrix, so system (8) has a unique equilibrium point if B is not a positive definite matrix. Now for system (5), zero also is an equilibrium point of system (5) since $f_{ij}(0) = 0$. Meanwhile, condition $uf_{ij}(u) > 0$ ($u \neq 0$) ensures that there exists a unique equilibrium point and it is exactly the zero point of system (5).

We adopt the following norms of vectors and matrices in this paper [22]: $\|u(t)\| = \sum_{i=1}^8 |u_i(t)|$, $\|B\| = \max_{1 \leq j \leq 8} \sum_{i=1}^8 |b_{ij}|$, the measure $\mu(B)$ of a matrix B is defined by $\mu(B) = \lim_{\theta \rightarrow 0^+} \frac{\|I + \theta B\| - 1}{\theta}$,

which for the chosen norms reduces to $\mu(B) = \max_{1 \leq j \leq 8} (b_{jj} + \sum_{i=1, i \neq j}^8 |b_{ij}|)$. $B > 0$ which indicates that B is a positive definite matrix.

Definition 1 The trivial solution of system (5) is unstable, if there exists at least one component of the trivial solution which is unstable.

Existence of periodic solutions

Theorem 1 Assume that system (5) has a unique equilibrium point for a set of selected parameters. If the following condition holds

$$9(\|B\|) \tau_*^2 \mu_* \exp(-\tau_* \mu_*) > 5 \quad (14)$$

where $\tau_* = \min \{\tau_1, \tau_2, \dots, \tau_8\}$, and $\mu_* = \min \{\mu_1, \mu_2, \dots, \mu_8\}$. Then the unique equilibrium point of the system (5) is unstable and it generates a limit cycle, namely, a periodic solution.

Proof We must prove that the unique equilibrium point of system (5) is unstable. Obviously, if the trivial solution of linearized system (7) or equivalent system (8) is unstable, then the trivial solution of system (5) is unstable. Thus we only need to consider system (8) and prove that the unique equilibrium point of system (8) which is exactly the zero point is unstable. Consider a special case of system (8) as follows:

$$U'(t) = AU(t) + B U(t - \tau_*) \quad (15)$$

where $\tau_* = \min \{\tau_1, \tau_2, \dots, \tau_8\}$, and $U(t - \tau_*) = [u_1(t - \tau_*), u_2(t - \tau_*), \dots, u_8(t - \tau_*)]^T$. Noting that $|u_i(t)| = u_i(t)$ as $u_i(t) > 0$ and $|u_i(t)| = -u_i(t)$ as $u_i(t) < 0$ ($i = 1, 2, \dots, 8$). From (15), when each $u_i(t) > 0$ we have

$$\frac{d|U(t)|}{dt} = AU(t) + B U(t - \tau_*) \quad (16)$$

and for each $u_i(t) < 0$ one can obtain

$$\frac{d|U(t)|}{dt} = -AU(t) - B U(t - \tau_*) \quad (17)$$

Noting that $-\mu_i < 0$ ($i = 1, 2, \dots, 8$). Therefore we have

$$\frac{d(\sum_{i=1}^8 |u_i(t)|)}{dt} \leq -\mu_* \sum_{i=1}^8 |u_i(t)| + \|B\| \sum_{i=1}^8 |u_i(t - \tau_*)| \quad (18)$$

Specially, for the scalar time delay differential equation

$$\frac{dz(t)}{dt} = -\mu_* z(t) + \|B\| z(t - \tau_*) \quad (19)$$

where $z(t) = \sum_{i=1}^8 |u_i(t)|$. If the unique equilibrium point of system (19) is stable, then the characteristic equation associated with (19) given by

$$\lambda = -\mu_* + \|B\| e^{-\lambda \tau_*} \quad (20)$$

will have a real negative root say λ_0 , and we have from

$$|\lambda_0| \geq \|B\| e^{|\lambda_0| \tau_*} - \mu_* \quad (21)$$

Using the formula $e^x \geq \frac{9}{5} x^2$ for $x \geq 0$ one can get

$$\begin{aligned} 1 &\geq \frac{\|B\| e^{|\lambda_0| \tau_*}}{\mu_* + |\lambda_0|} = \frac{\|B\| \tau_* e^{-\mu_* \tau_*} e^{(\mu_* + |\lambda_0|) \tau_*}}{(\mu_* + |\lambda_0|) \tau_*} \geq (\|B\| \tau_* e^{-\mu_* \tau_*}) \frac{9(\mu_* + |\lambda_0|) \tau_*}{5} \\ &\geq (\|B\| \tau_* e^{-\mu_* \tau_*}) \frac{9\mu_* \tau_*}{5} = \frac{9}{5} (\|B\|) \tau_*^2 \mu_* \exp(-\tau_* \mu_*) \end{aligned} \quad (22)$$

The last inequality contradicts (14). Hence, our claim regarding the instability of the equilibrium point of system (15) is valid. Based on the comparison theorem of differential equation, we have $\sum_{i=1}^8 |u_i(t)| \leq z(t)$. According to the definition of the instability of the trivial

solution, for an arbitrary $\varepsilon > 0$, there exists a sequence $\{t_k\}$ such that $|z(t_k)| > \varepsilon$, where $z(t)$ represents the trivial solution of system (15). Since $\sum_{i=1}^8 |u_i(t)| \leq z(t)$, this means that there exists a subsequence $\{t_{k_j}\}$ of the sequence $\{t_k\}$ such that $\sum_{i=1}^8 |u_i(t_{k_j})| \leq z(t_{k_j})$. Therefore, there exists at least one $u_i(t)$, and without loss of generality, we assume that $|u_1(t_{k_j})| > \frac{\varepsilon}{8}$. Since ε is an arbitrary sufficiently small positive number, $\frac{\varepsilon}{8}$ is also an arbitrary sufficiently small positive number. Thus, $u_1(t)$ is unstable. According to the definition 1, the instability of the component $u_1(t)$ implies that the trivial solution of system (15) is unstable.

Now we prove that the trivial solution of system (9) is also unstable. System (15) is a special case of (9). Obviously, $u_i(t - \tau_*)$ is equivalent to $u_i(t - \tau_i)$ ($i = 1, 2, \dots, 8$) as t is sufficiently large. So, we still have

$|u_1(t_{k_j})| > \frac{\varepsilon}{8}$ as t_{k_j} is sufficiently large. This means that the trivial solution of system (9) is unstable. We can then prove that the trivial solution of system (5) is unstable. Indeed, system (9) is a linearized version of system (5), in other words, system (5) is a disturbing system of (9).

Thus, the instability of the trivial solution of system (9) implies that the instability of the trivial solution of the original system (5). Since all solutions of system (5) are bounded, the instability of the unique equilibrium point together with boundedness of the solutions lead system (5) to generate a limit cycle, namely, a periodic solution based on Chafee's criterion [23] and the appendix of [24].

Theorem 2 Assume that system (5) has a unique equilibrium point for a set of selected parameters. If the following condition holds

$$\|B\| - \mu_* > 0 \quad (23)$$

then the unique equilibrium point of the system (5) is unstable. System (5) generates a limit cycle, namely, a periodic solution.

Proof We still prove that the trivial solution of system (15) is unstable. The characteristic equation of system (15) is the following

$$\lambda = -\mu_* + \|B\| e^{-\lambda\tau_*} \quad (24)$$

namely

$$\lambda + \mu_* - \|B\| e^{-\lambda\tau_*} = 0 \quad (25)$$

and there exists a positive characteristic root of equation (25) under the restrictive condition (23). Indeed, let $f(\lambda) = \lambda + \mu_* - \|B\| e^{-\lambda\tau_*}$, then $f(\lambda)$ is a continuous function of λ . Obviously, $f(0) = \mu_* - \|B\| = -(\|B\| - \mu_*) < 0$, and while as $\lambda (> 0)$ is sufficiently large, $e^{-\lambda\tau_*}$ will be sufficiently small, Therefore,

there exists a $\bar{\lambda} (> 0)$ such that $f(\bar{\lambda}) = \bar{\lambda} + \mu_* - \|B\| e^{-\bar{\lambda}\tau_*} > 0$. According to the well known the Intermediate Value Theorem, there exists a positive value of λ say λ_1 , $\lambda_1 \in (0, \bar{\lambda})$ such that $f(\lambda_1) = \lambda_1 + \mu_* - \|B\| e^{-\lambda_1\tau_*} = 0$. In other words, equation (25) has a positive characteristic root. Therefore, the trivial solution of system (15) is unstable. Similar to the proof of Theorem 1, the unique equilibrium point of system (5) is unstable. According to the generalized Chafee's criterion, system (5) has a limit cycle, namely, a periodic solution.

Simulation result

These simulations are performed by using the system (5). First, the activation function is selected as $f_{ij}(z) = \tanh(z)$. Then $f'(z) = 1 - \tanh^2(z)$, so $f'(0) = 1$. The parameters are selected as $\mu_1 = 0.45$,

$$\mu_2 = 0.64, \mu_3 = 0.48, \mu_4 = 0.55, \mu_5 = 0.62, \mu_6 = 0.58, \mu_7 = 0.52, \mu_8 = 0.65; c_{11} = 1.25, c_{12} = 1.15,$$

$$c_{13} = 0.85, c_{14} = 0.75, c_{21} = -2.15, c_{22} = 0.55, c_{23} = -1.05, c_{24} = 0.45, c_{31} = 1.35, c_{32} = -0.48,$$

$$c_{33} = 0.52, c_{34} = -0.25, c_{41} = 0.75, c_{42} = 1.25, c_{43} = -0.35, c_{44} = 0.36, c_{51} = -0.95, c_{52} = 0.85,$$

$$c_{53} = -1.16, c_{54} = 0.88, c_{61} = -2.65, c_{62} = -0.75, c_{63} = 0.82, c_{64} = 0.45, c_{71} = -0.45, c_{72} = -0.32, c_{73} = 0.64, c_{74} = -0.25, c_{81} = -0.42, c_{82} = 0.36, c_{83} = -0.12, c_{84} = -0.85; \text{time delays are selected as } \tau_1 = 1.12, \tau_2 = 1.16, \tau_3 = 1.25, \tau_4 = 1.22, \tau_5 = 1.15, \tau_6 = 1.24, \tau_7 = 1.26, \tau_8 = 1.18. \text{ The characteristic values of matrix } B \text{ are } -0.7177, 0.7177, -0.3376 \pm 1.9017 i, 0.3376 \pm 1.9017 i, \pm 0.1104 i. \text{ Thus, } B \text{ is not a positive definite matrix. From Lemma 2, this system has a unique equilibrium point, namely, the zero point. We have } \tau_* = 1.12, \mu_* = 0.45, \|B\| = 5.5 \text{ and } 9(\|B\|) \tau_*^2 \mu_* \exp(-\tau_* \mu_*) = 9 \times 5.5 \times 1.12 \times 1.12 \times 0.45 \times \exp(-1.12 \times 0.45) = 16.8799 > 5.$$

Based on Theorem 1, there is a periodic solution (see figure 1). Then we change time delays as $\tau_1 = 0.52, \tau_2 = 0.56, \tau_3 = 0.58, \tau_4 = 0.62, \tau_5 = 0.65, \tau_6 = 0.64, \tau_7 = 0.46, \tau_8 = 0.48$, the other parameters are the same as in figure 1. Obviously, the conditions of Theorem 2 are satisfied, there exists a periodic solution (see figure 2). However, we see that $9(\|B\|) \tau_*^2 \mu_* \exp(-\tau_* \mu_*) = 9 \times 5.5 \times 0.46 \times 0.46 \times 0.45 \times \exp(-0.46 \times 0.45) = 3.8321 < 5$.

The condition of Theorem 1 is not satisfied. This means that the Theorem 1 is a stronger sufficient condition. Then we select the activation function as $f_{ij}(z) = \arctan(z)$. Keeping all parameters similar to those used to generate figure 2, we see that the oscillatory frequency and amplitude remain almost the same (see figure 3), implying that the oscillatory behavior is just a little effected by the activation functions. However, when we increase the time delays, the oscillatory frequency changes greatly (see figure 4). In order to see the effect of μ_i , we change $\mu_1 = 1.25, \mu_2 = 1.42, \mu_3 = 1.48, \mu_4 = 1.35, \mu_5 = 1.42, \mu_6 = 1.52, \mu_7 = 1.46, \mu_8 = 1.38$, the other parameters are the same as in figure 4, time delays are selected as $\tau_1 = 0.72, \tau_2 = 0.76, \tau_3 = 0.78, \tau_4 = 0.82, \tau_5 = 0.85, \tau_6 = 0.84, \tau_7 = 0.86, \tau_8 = 0.88$, and $\tau_1 = 1.52, \tau_2 = 1.56, \tau_3 = 1.58, \tau_4 = 1.62, \tau_5 = 1.65, \tau_6 = 1.64, \tau_7 = 1.46, \tau_8 = 1.48$, respectively. We see that the oscillatory frequency changes very much (see figure 5 and figure 6). This means that time delays and μ_i effect the oscillatory frequency. Finally we select another set of parameters as $\mu_1 = 0.85, \mu_2 = 0.82, \mu_3 = 0.75, \mu_4 = 0.72, \mu_5 = 0.82, \mu_6 = 0.78, \mu_7 = 0.76, \mu_8 = 0.84; c_{11} = -0.45, c_{12} = 1.25, c_{13} = 0.35, c_{14} = 0.55, c_{21} = -1.65, c_{22} = 0.85, c_{23} = 1.45, c_{24} = 0.75, c_{31} = 1.15, c_{32} = -0.68, c_{33} = 0.22, c_{34} = -0.58, c_{41} = 0.15, c_{42} = 1.65, c_{43} = -0.95, c_{44} = 0.46, c_{51} = -0.75, c_{52} = 0.45, c_{53} = -1.76, c_{54} = 0.38, c_{61} = -2.35, c_{62} = -0.92, c_{63} = 0.42, c_{64} = 0.75, c_{71} = -0.85, c_{72} = -0.72, c_{73} = 0.24, c_{74} = -0.85, c_{81} = -0.32, c_{82} = 0.96, c_{83} = -0.42, c_{84} = -0.15. We see that the characteristic values of matrix B are $-1.2444, 1.2444, \pm 2.2159 i, \pm 1.2138 i, \pm 0.2593 i$, and $\|B\| = 4.27$. Therefore, B is not a positive definite matrix. From Lemma 2, the system has a unique equilibrium point, namely, the zero point. When time delays are selected as $\tau_1 = 0.85, \tau_2 = 0.82, \tau_3 = 0.75, \tau_4 = 0.72, \tau_5 = 0.82, \tau_6 = 0.78, \tau_7 = 0.76, \tau_8 = 0.80$, and time delays are selected as $\tau_1 = 1.24, \tau_2 = 1.26, \tau_3 =$$

1.28, $\tau_4 = 1.32$, $\tau_5 = 1.35$, $\tau_6 = 1.34$, $\tau_7 = 1.38$, $\tau_8 = 1.36$, respectively, there exist periodic oscillations based on the Theorem 2 (see figure 7 and figure 8).

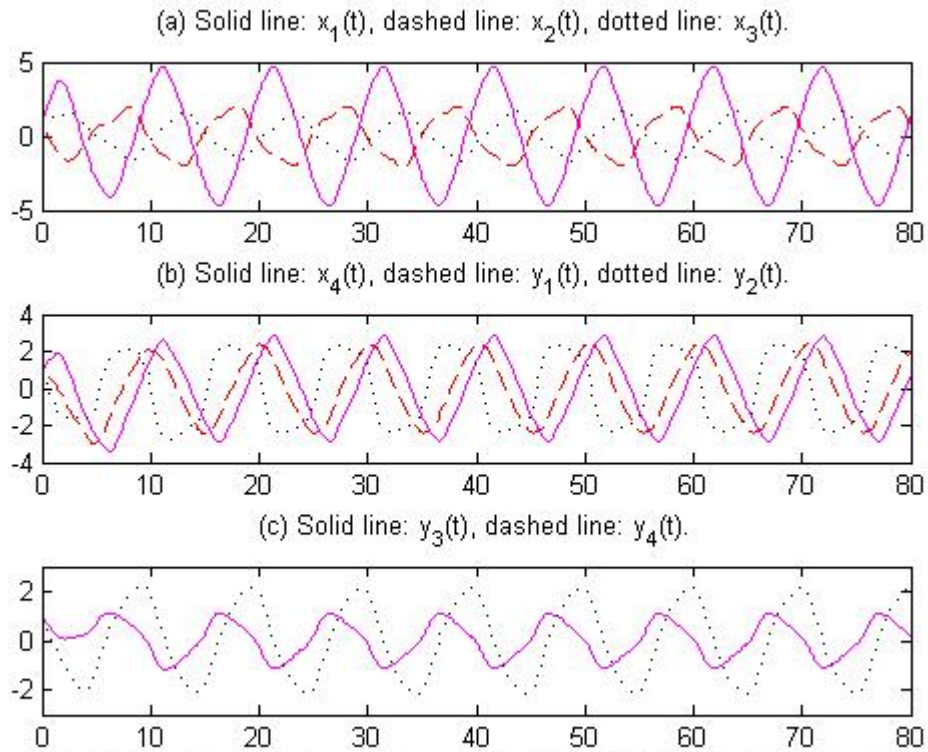


Fig. 1 Oscillation of the solutions, delays: 1.12, 1.16, 1.25, 1.22, 1.15, 1.24, 1.26, 1.18.

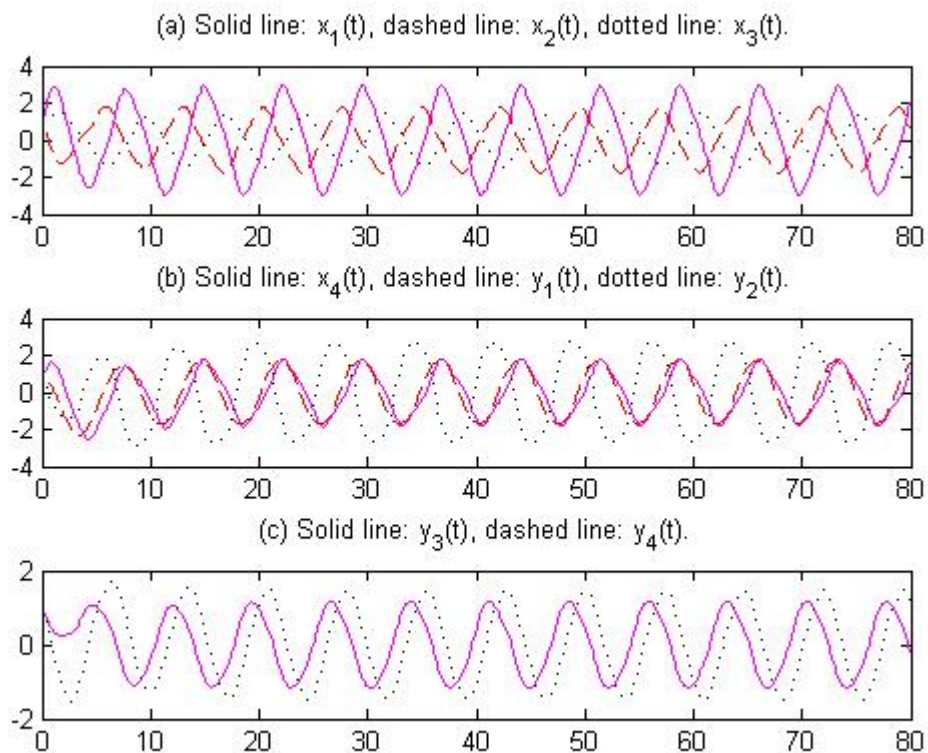


Fig. 2 Oscillation of the solutions, delays: 0.52, 0.56, 0.58, 0.62, 0.65, 0.64, 0.46, 0.48.

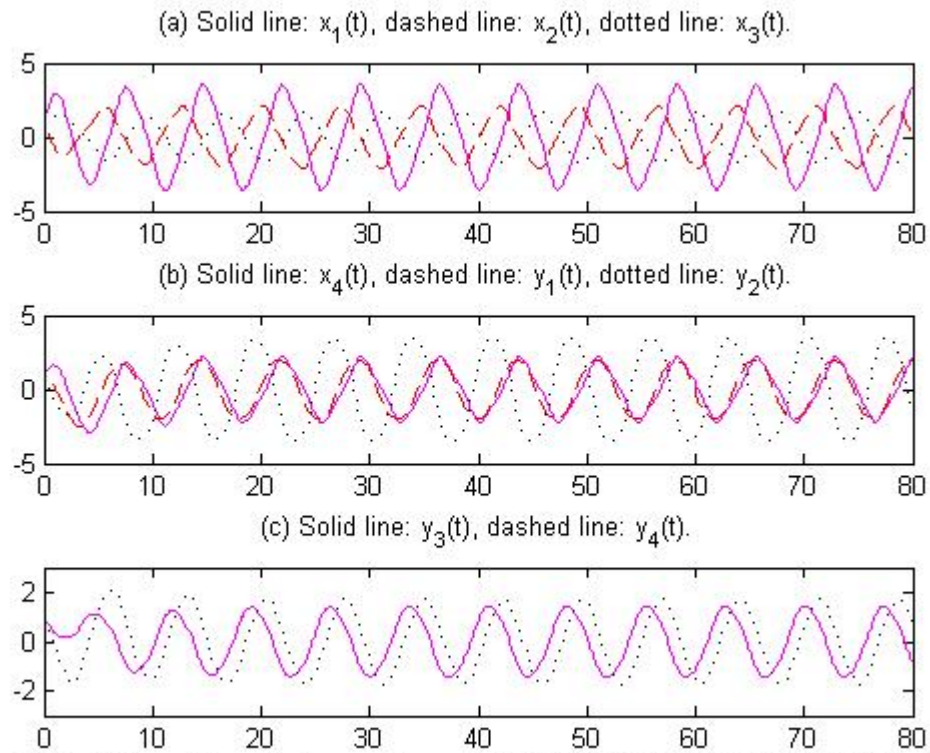


Fig. 3 Oscillation of the solutions, delays: 0.52, 0.56, 0.58, 0.62, 0.65, 0.64, 0.46, 0.48.

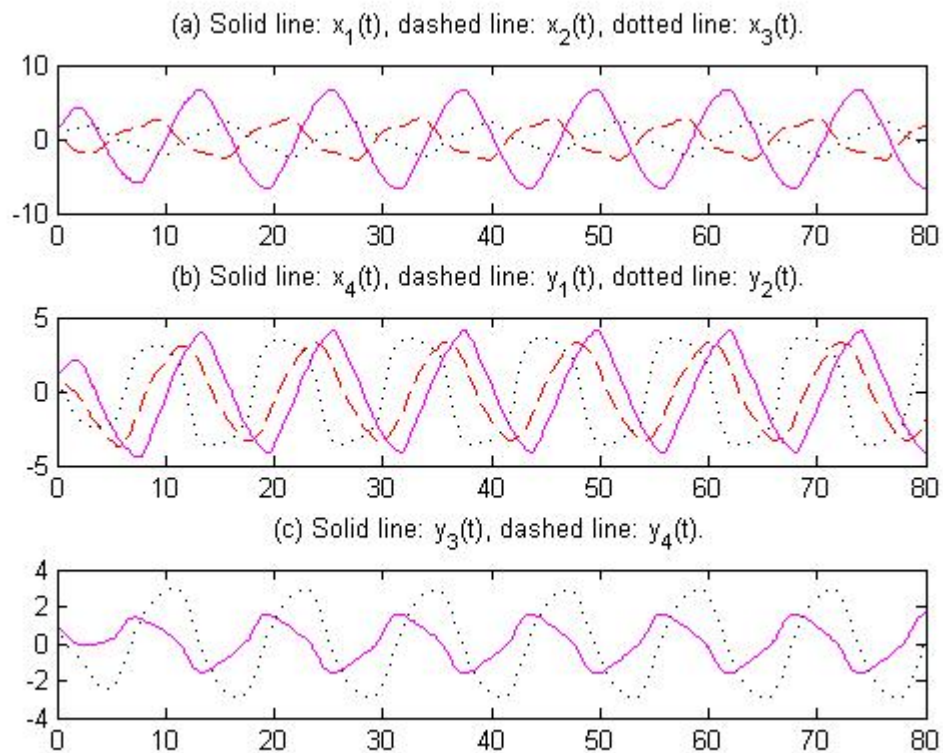


Fig. 4 Oscillation of the solutions, delays: 1.52, 1.56, 1.58, 1.62, 1.65, 1.64, 1.46, 1.48.

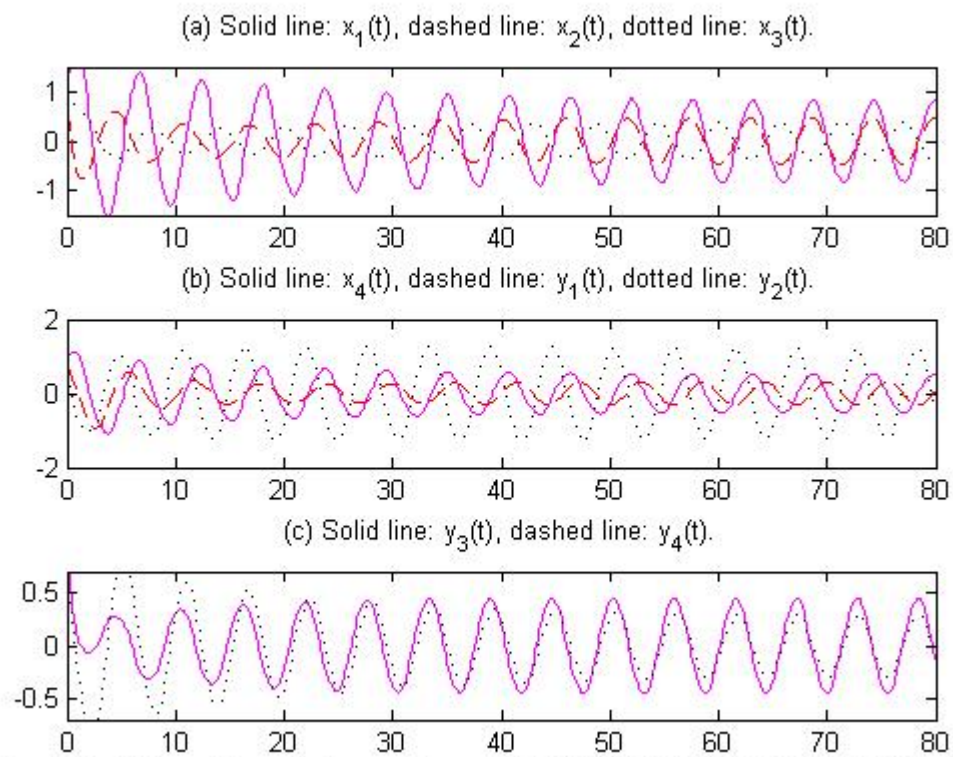


Fig. 5 Oscillation of the solutions: delays: 0.72, 0.76, 0.78, 0.82, 0.85, 0.84, 0.86, 0.88.

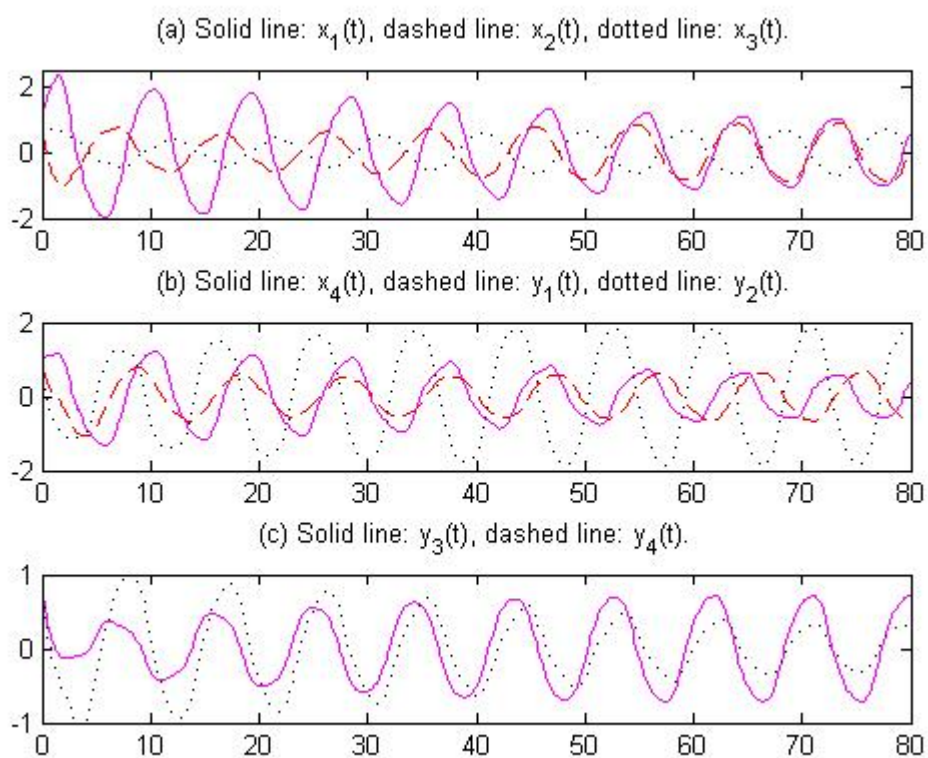


Fig. 6 Oscillation of the solutions, delays: 1.52, 1.56, 1.58, 1.62, 1.65, 1.64, 1.46, 1.48.

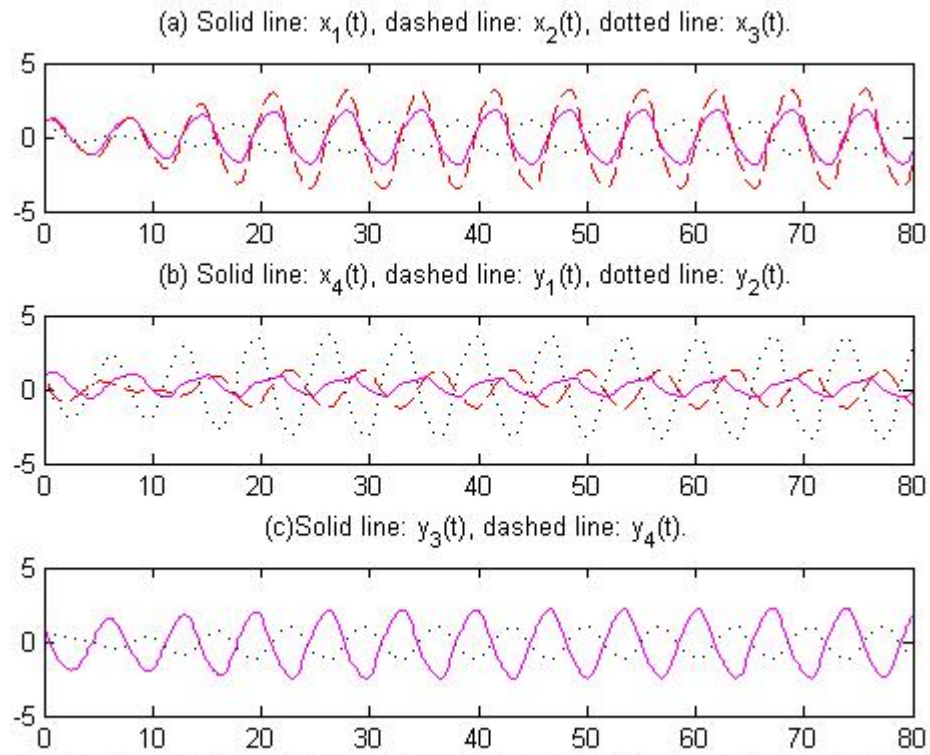


Fig. 7 Oscillation of the solutions, delays: 0.85, 0.82, 0.75, 0.72, 0.82, 0.78, 0.76, 0.8.

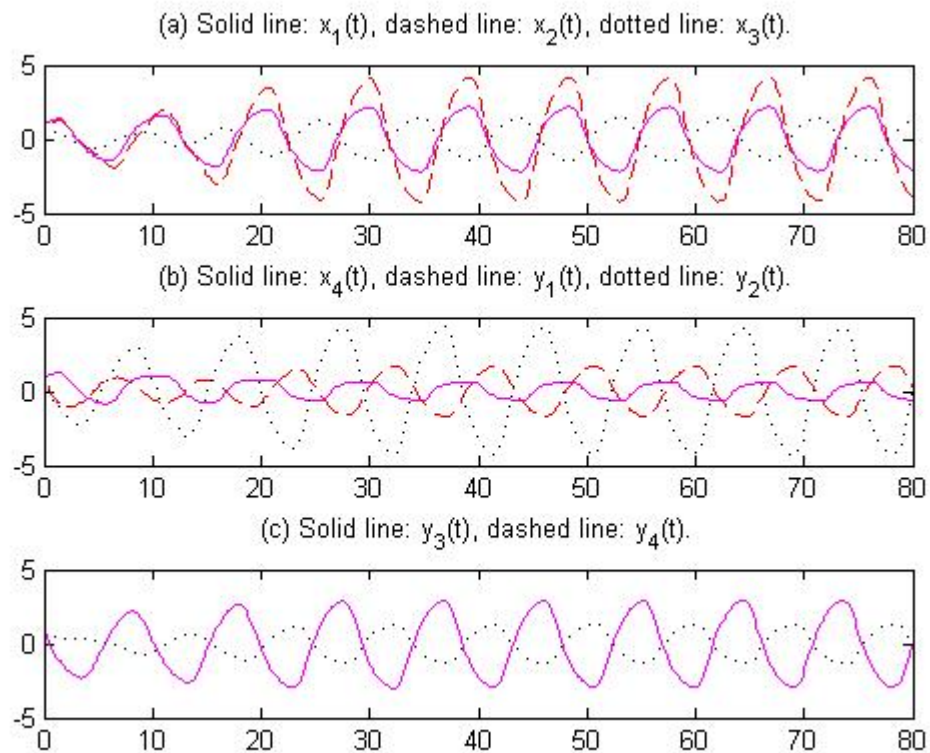


Fig. 8 Oscillation of the solutions, delays: 1.24, 1.26, 1.28, 1.32, 1.35, 1.34, 1.38, 1.36.

CONCLUSION

This paper considers an eight-neuron BAM network with discrete delays. Two sufficient conditions to guarantee the existence of periodic solutions are provided. A specific selection of parameters is used to demonstrate the results. Our simple criteria to ensure the existence of permanent oscillations are easy to check, as compared to predicting the regions of bifurcation. The computer simulation suggests that our theorems are only sufficient conditions.

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