

Periodic oscillation for a complex-valued neural network model with discrete and distributed delays

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Abstract: In this paper, a complex-valued neural network model with discrete and distributed delays is investigated under the assumption that the activation function can be separated into its real and imaginary parts. Based on the mathematical analysis method, some sufficient conditions to guarantee the existence of periodic oscillatory solutions are established. Computer simulation is given to illustrate the validity of the theoretical results.

Keywords: complex-valued neural network model, discrete delay, distributed delay, periodic oscillation

1 Introduction

In the past two decades, many researchers have studied various complex-valued neural network models with or without discrete and distributed delays [1-20]. For example, Samidurai *et al.* have discussed the delay-dependent stability for some complex-valued neural networks with discrete and distributed time-varying delays [1,2]. Wang *et al.* considered the stability for complex-valued stochastic delayed networks with Markovian switching and impulsive effects [3]. Guo *et al.* studied the existence, uniqueness, and exponential stability for complex-valued memristor-based BAM neural networks with time delays [4]. Popa discussed the global μ -stability of neutral-type impulsive complex-valued BAM neural networks with leakage delay and unbounded time-varying delays [5]. Many authors focus on the synchronization of complex-valued networks, such as exponential synchronization of complex-valued delayed coupled systems on networks with aperiodically on-off coupling [7,8]. Synchronization of impulsive coupled complex-valued neural networks with delay by the matrix measure method, distributed impulsive method, a direct error method, and a non-reduced order and non-separation approach [9, 10, 11, 12]. With complex-valued neural networks, a new multi-modal technique for

assisting visually-impaired people in recognizing objects in public indoor environment was provided [16]. For a complex-valued Wilson-Cowan neural network with time delay as follows:

$$\begin{cases} v_1'(t) = -v_1(t) + a_1g(v_1(t)) + a_2g(v_2(t - \tau)) + P, \\ v_2'(t) = -v_2(t) + a_3g(v_1(t - \tau)) + a_4g(v_2(t)) + Q. \end{cases} \quad (1)$$

By using proper translations and coordinate transformations, Ji *et al.* have decomposed the activation functions and connection weights into their real and imaginary parts, so as to construct an equivalent real-valued system. Then, the sufficient conditions for Hopf bifurcation and its directions are provided through normal form theory and central manifold theorem [19].

Hang *et al.* have investigated a two-node fractional complex-valued neural network with time delay as follows [20]:

$$\begin{cases} D^q z_1'(t) = -\mu_1 z_1(t) + af(z_1(t - \tau)) + bf(z_2(t - \tau)), \\ D^q z_2'(t) = -\mu_2 z_2(t) + cf(z_1(t - \tau)) + df(z_2(t - \tau)). \end{cases} \quad (2)$$

By using time delay as the bifurcation parameter, the dynamical behaviors that including local asymptotical stability and Hopf bifurcation were discussed, the conditions of emergence of bifurcation were also obtained. Furthermore, it reveals that the onset of the bifurcation point can be delayed as the order increases. In [21], Li *et al.* extended a real-valued network model to a complex-valued model with discrete and distributed delays as the following:

$$\begin{cases} z_1'(t) = -z_1(t) + b_{11}f_{11}(z_1(t - \tau)) + b_{12}f_{12}(\int_{-\infty}^t F(t - s)z_2(s)ds), \\ z_2'(t) = -z_2(t) + b_{21}f_{21}(z_1(t - \tau)) + b_{22}f_{22}(\int_{-\infty}^t F(t - s)z_2(s)ds). \end{cases} \quad (3)$$

Regarding the discrete time delay as the bifurcating parameter, the problem of Hopf bifurcation in the newly-proposed complex-valued neural network model was investigated under the assumption that the activation function can be separated into its real and imaginary parts. Based on the normal form theory and center manifold theorem, some sufficient conditions which determine the direction of the Hopf bifurcation and the stability of the bifurcating periodic solutions were established. Zhang *et al.* have considered a complex value delayed Hopfield neural networks model, in which a ring topology consists of two coupling unidirectional rings, each with four oscillators is given [22]. For a real-valued three-node network as follows:

$$\begin{cases} x_1'(t) = -x_1(t) + a \int_{-\infty}^t k(t - s)f(x_1(s))ds + b[f(x_3(t - \tau)) + f(x_2(t - \tau))], \\ x_2'(t) = -x_2(t) + a \int_{-\infty}^t k(t - s)f(x_2(s))ds + b[f(x_1(t - \tau)) + f(x_3(t - \tau))], \\ x_3'(t) = -x_3(t) + a \int_{-\infty}^t k(t - s)f(x_3(s))ds + b[f(x_2(t - \tau)) + f(x_1(t - \tau))]. \end{cases} \quad (4)$$

In this model, the distributed signal transmission delay has introduced in the self-connection of the network [23]. The author has discussed the Hopf bifurcation and stability switching of model (4).

Motivated by the above models, In this paper, we extend model (4) to the following complex-valued network system:

$$\begin{cases} z_1'(t) = -l_1 z_1(t) + r_{11} f_{11}(\int_{-\infty}^t F(t-s) z_1(s) ds) + r_{12} f_{12}(z_2(t-\tau)) + r_{13} f_{13}(z_3(t-\tau)), \\ z_2'(t) = -l_2 z_2(t) + r_{21} f_{21}(z_1(t-\tau)) + r_{22} f_{22}(\int_{-\infty}^t F(t-s) z_2(s) ds) + r_{23} f_{23}(z_3(t-\tau)), \\ z_3'(t) = -l_3 z_3(t) + r_{31} f_{31}(z_1(t-\tau)) + r_{32} f_{32}(z_2(t-\tau)) + r_{33} f_{33}(\int_{-\infty}^t F(t-s) z_3(s) ds), \end{cases} \quad (5)$$

where $F(s) = \alpha e^{-\alpha s}$, $s \geq 0, \alpha > 0$ is a weak kernel. $l_j, r_{kj} (k, j = 1, 2, 3)$ are complex numbers. Let $z_4(t) = \int_{-\infty}^t F(t-s) z_1(s) ds$, $z_5(t) = \int_{-\infty}^t F(t-s) z_2(s) ds$, $z_6(t) = \int_{-\infty}^t F(t-s) z_3(s) ds$, then system (5) changes to the following

$$\begin{cases} z_1'(t) = -l_1 z_1(t) + r_{11} f_{11}(z_4(t)) + r_{12} f_{12}(z_2(t-\tau)) + r_{13} f_{13}(z_3(t-\tau)), \\ z_2'(t) = -l_2 z_2(t) + r_{21} f_{21}(z_1(t-\tau)) + r_{22} f_{22}(z_5(t)) + r_{23} f_{23}(z_3(t-\tau)), \\ z_3'(t) = -l_3 z_3(t) + r_{31} f_{31}(z_1(t-\tau)) + r_{32} f_{32}(z_2(t-\tau)) + r_{33} f_{33}(z_6(t)), \\ z_4'(t) = -\alpha z_4(t) + \alpha z_1(t), \\ z_5'(t) = -\alpha z_5(t) + \alpha z_2(t), \\ z_6'(t) = -\alpha z_6(t) + \alpha z_3(t), \end{cases} \quad (6)$$

It was emphasized that the bifurcating method is hard to deal with system (6). From [21] one can see that the authors need to investigate a 6-degree bifurcating equation (see equation (11), page 155). Similarly we must consider a 12-degree bifurcating equation according to the bifurcating method. It is quite not easy to deal with a 12-degree algebraic equation. Therefore, in this paper, by means of the mathematical analysis method, we discuss the periodic oscillation for system (6). For convenience, let $l_j = a_j + ib_j, r_{kj} = p_{kj} + iq_{kj}, f_{kj}(z_j(t)) = f_{kj}^R(x_j(t), y_j(t)) + if_{kj}^I(x_j(t), y_j(t)) (z_j(t) = x_j(t) + iy_j(t)), k, j = 1, 2, 3$. Then the complex-valued system (6) can be expressed by separating it into real

and imaginary parts as the following:

$$\left\{ \begin{array}{l}
 x'_1(t) = -a_1x_1(t) + b_1y_1(t) + p_{11}f_{11}^R(x_4(t), y_4(t)) - q_{11}f_{11}^I(x_4(t), y_4(t)) \\
 \quad + p_{12}f_{12}^R(x_2(t-\tau), y_2(t-\tau)) - q_{12}f_{12}^I(x_2(t-\tau), y_2(t-\tau)) \\
 \quad + p_{13}f_{13}^R(x_3(t-\tau), y_3(t-\tau)) - q_{13}f_{13}^I(x_3(t-\tau), y_3(t-\tau)), \\
 y'_1(t) = -a_1y_1(t) - b_1x_1(t) + p_{11}f_{11}^I(x_4(t), y_4(t)) + q_{11}f_{11}^R(x_4(t), y_4(t)) \\
 \quad + p_{12}f_{12}^I(x_2(t-\tau), y_2(t-\tau)) + q_{12}f_{12}^R(x_2(t-\tau), y_2(t-\tau)) \\
 \quad + p_{13}f_{13}^I(x_3(t-\tau), y_3(t-\tau)) + q_{13}f_{13}^R(x_3(t-\tau), y_3(t-\tau)), \\
 x'_2(t) = -a_2x_2(t) + b_2y_2(t) + p_{21}f_{21}^R(x_1(t-\tau), y_1(t-\tau)) - q_{21}f_{21}^I(x_1(t-\tau), y_1(t-\tau)) \\
 \quad + p_{22}f_{22}^R(x_5(t), y_5(t)) - q_{22}f_{22}^I(x_5(t), y_5(t)) \\
 \quad + p_{23}f_{23}^R(x_3(t-\tau), y_3(t-\tau)) - q_{23}f_{23}^I(x_3(t-\tau), y_3(t-\tau)), \\
 y'_2(t) = -a_2y_2(t) - b_2x_2(t) + p_{21}f_{21}^I(x_1(t-\tau), y_1(t-\tau)) + q_{21}f_{21}^R(x_1(t-\tau), y_1(t-\tau)) \\
 \quad + p_{22}f_{22}^I(x_5(t), y_5(t)) + q_{22}f_{22}^R(x_5(t), y_5(t)) \\
 \quad + p_{23}f_{23}^I(x_3(t-\tau), y_3(t-\tau)) + q_{23}f_{23}^R(x_3(t-\tau), y_3(t-\tau)), \\
 x'_3(t) = -a_3x_3(t) + b_3y_3(t) + p_{31}f_{31}^R(x_1(t-\tau), y_1(t-\tau)) - q_{31}f_{31}^I(x_1(t-\tau), y_1(t-\tau)) \\
 \quad + p_{32}f_{32}^R(x_2(t-\tau), y_2(t-\tau)) - q_{32}f_{32}^I(x_2(t-\tau), y_2(t-\tau)) \\
 \quad + p_{33}f_{33}^R(x_6(t), y_6(t)) - q_{33}f_{33}^I(x_6(t), y_6(t)), \\
 y'_3(t) = -a_3y_3(t) - b_3x_3(t) + p_{31}f_{31}^I(x_1(t-\tau), y_1(t-\tau)) + q_{31}f_{31}^R(x_1(t-\tau), y_1(t-\tau)) \\
 \quad + p_{32}f_{32}^I(x_2(t-\tau), y_2(t-\tau)) + q_{32}f_{32}^R(x_2(t-\tau), y_2(t-\tau)) \\
 \quad + p_{33}f_{33}^I(x_6(t), y_6(t)) + q_{33}f_{33}^R(x_6(t), y_6(t)), \\
 x'_4(t) = -\alpha x_4(t) + \alpha x_1(t), \\
 y'_4(t) = -\alpha y_4(t) + \alpha y_1(t), \\
 x'_5(t) = -\alpha x_5(t) + \alpha x_2(t), \\
 y'_5(t) = -\alpha y_5(t) + \alpha y_2(t), \\
 x'_6(t) = -\alpha x_6(t) + \alpha x_3(t), \\
 y'_6(t) = -\alpha y_6(t) + \alpha y_3(t).
 \end{array} \right. \quad (7)$$

Thus, in order to discuss the periodic oscillation of system (6), we only need to consider the periodic oscillation of system (7). Suppose that the derivative of $f_{kj}^R(x, y)$ and $f_{kj}^I(x, y)$ with respect to x and

y exist, continuous, and $f_{kj}^R(0,0) = 0, f_{kj}^I(0,0) = 0$. Then the linearized system of (6) is the following:

$$\left\{ \begin{array}{l} x_1'(t) = -a_1x_1(t) + b_1y_1(t) + c_{14}x_4(t) + d_{14}y_4(t) + c_{12}x_2(t-\tau) + d_{12}y_2(t-\tau) \\ \quad + c_{13}x_3(t-\tau) + d_{13}y_3(t-\tau)), \\ y_1'(t) = -a_1y_1(t) - b_1x_1(t) + m_{14}x_4(t) + n_{14}y_4(t) + m_{12}x_2(t-\tau) + n_{12}y_2(t-\tau) \\ \quad + m_{13}x_3(t-\tau) + n_{13}y_3(t-\tau)), \\ x_2'(t) = -a_2x_2(t) + b_2y_2(t) + c_{25}x_5(t) + d_{25}y_5(t) + c_{21}x_1(t-\tau) + d_{21}y_1(t-\tau) \\ \quad + c_{23}x_3(t-\tau) + d_{23}y_3(t-\tau)), \\ y_2'(t) = -a_2y_2(t) - b_2x_2(t) + m_{25}x_5(t) + n_{25}y_5(t) + m_{21}x_1(t-\tau) + n_{21}y_1(t-\tau) \\ \quad + m_{23}x_3(t-\tau) + n_{23}y_3(t-\tau)), \\ x_3'(t) = -a_3x_3(t) + b_3y_3(t) + c_{36}x_6(t) + d_{36}y_6(t) + c_{31}x_1(t-\tau) + d_{31}y_1(t-\tau) \\ \quad + c_{32}x_2(t-\tau) + d_{32}y_2(t-\tau)), \\ y_3'(t) = -a_3y_3(t) - b_3x_3(t) + m_{36}x_6(t) + n_{36}y_6(t) + m_{31}x_1(t-\tau) + n_{31}y_1(t-\tau) \\ \quad + m_{32}x_2(t-\tau) + n_{32}y_2(t-\tau)), \\ x_4'(t) = -\alpha x_4(t) + \alpha x_1(t), \\ y_4'(t) = -\alpha y_4(t) + \alpha y_1(t), \\ x_5'(t) = -\alpha x_5(t) + \alpha x_2(t), \\ y_5'(t) = -\alpha y_5(t) + \alpha y_2(t), \\ x_6'(t) = -\alpha x_6(t) + \alpha x_3(t), \\ y_6'(t) = -\alpha y_6(t) + \alpha y_3(t). \end{array} \right. \quad (8)$$

where for $k=1,2,3, j = 4,5,6$, $c_{kj} = p_{kk} \frac{\partial f_{kk}^R(0,0)}{\partial x_j} - q_{kk} \frac{\partial f_{kk}^I(0,0)}{\partial x_j}$, $d_{kj} = p_{kk} \frac{\partial f_{kk}^R(0,0)}{\partial y_j} - q_{kk} \frac{\partial f_{kk}^I(0,0)}{\partial y_j}$, $m_{kj} = p_{kk} \frac{\partial f_{kk}^I(0,0)}{\partial x_j} + q_{kk} \frac{\partial f_{kk}^R(0,0)}{\partial x_j}$, $n_{kj} = p_{kk} \frac{\partial f_{kk}^I(0,0)}{\partial y_j} + q_{kk} \frac{\partial f_{kk}^R(0,0)}{\partial y_j}$. For $k, j=1,2,3$, $c_{kj} = p_{kj} \frac{\partial f_{kj}^R(0,0)}{\partial x_j} - q_{kj} \frac{\partial f_{kj}^I(0,0)}{\partial x_j}$, $d_{kj} = p_{kj} \frac{\partial f_{kj}^R(0,0)}{\partial y_j} - q_{kj} \frac{\partial f_{kj}^I(0,0)}{\partial y_j}$, $m_{kj} = p_{kj} \frac{\partial f_{kj}^I(0,0)}{\partial x_j} + q_{kj} \frac{\partial f_{kj}^R(0,0)}{\partial x_j}$, $n_{kj} = p_{kj} \frac{\partial f_{kj}^I(0,0)}{\partial y_j} + q_{kj} \frac{\partial f_{kj}^R(0,0)}{\partial y_j}$. The matrix form of system (8) is the following:

$$U'(t) = AU(t) + BU(t-\tau) \quad (9)$$

where $U(t) = [x(t), y(t), \dots, x(t), y(t)]$, $U(t-\tau) = [x(t-\tau), y(t-\tau), x(t-\tau), y(t-\tau), x_3(t-\tau), y_3(t-\tau), 0, 0, 0, 0, 0, 0]^T$, Both $A = (a_{ij})_{12 \times 12}$ and $B = (b_{ij})_{12 \times 12}$ are 12×12 matrices as follows:

$$A = (a_{ij})_{12 \times 12} = \begin{pmatrix} -a_1 & b_1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -b_1 & -a_1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & -a_2 & b_2 & 0 & \cdots & d_{25} & 0 & 0 \\ 0 & 0 & -b_2 & -a_2 & 0 & \cdots & n_{25} & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \alpha & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha & 0 & \cdots & -\alpha & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha & \cdots & 0 & -\alpha & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & -\alpha \end{pmatrix},$$

$$B = (b_{ij})_{12 \times 12} = \begin{pmatrix} 0 & 0 & c_{12} & d_{12} & c_{13} & \cdots & 0 & 0 & 0 \\ 0 & 0 & m_{12} & n_{12} & m_{13} & \cdots & 0 & 0 & 0 \\ c_{21} & d_{21} & 0 & 0 & c_{23} & \cdots & 0 & 0 & 0 \\ m_{21} & n_{21} & 0 & 0 & m_{23} & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix}.$$

2 Preliminaries

Let C is a six by six matrix as follows:

$$C = (c_{ij})_{6 \times 6} = \begin{pmatrix} p_{11} & -q_{11} & p_{12} & -q_{12} & p_{13} & -q_{13} \\ q_{11} & p_{11} & q_{12} & p_{12} & q_{13} & p_{13} \\ p_{21} & -q_{21} & p_{22} & -q_{22} & p_{23} & -q_{23} \\ q_{21} & p_{21} & q_{22} & p_{22} & q_{23} & p_{23} \\ p_{31} & -q_{31} & p_{32} & -q_{32} & p_{33} & -q_{33} \\ q_{31} & p_{31} & q_{32} & p_{32} & q_{33} & p_{33} \end{pmatrix}.$$

Lemma 1 Assume that $a_j > 0, b_j > 0 (j = 1, 2, 3)$, $f_{jk}^R(0, 0) = 0, f_{jk}^I(0, 0) = 0, f_{jk}^R(x, y) > 0, f_{jk}^I(x, y) > 0$ when $x > 0, y > 0$, while $f_{jk}^R(x, y) < 0, f_{jk}^I(x, y) < 0$ when $x < 0, y < 0 (j, k = 1, 2, 3)$, C is not a positive definite matrix, then system (7) has a unique equilibrium.

Proof An equilibrium point $x^* = [x_1^*, y_1^*, \dots, x_6^*, y_6^*]^T$ of system (7) is a constant solution of the

following algebraic equation

$$\left\{ \begin{array}{l} -a_1x_1^* + b_1y_1^* + p_{11}f_{11}^R(x_4^*, y_4^*) - q_{11}f_{11}^I(x_4^*, y_4^*) + p_{12}f_{12}^R(x_2^*, y_2^*) \\ \quad - q_{12}f_{12}^I(x_2^*, y_2^*) + p_{13}f_{13}^R(x_3^*, y_3^*) - q_{13}f_{13}^I(x_3^*, y_3^*) = 0, \\ -a_1y_1^* - b_1x_1^* + p_{11}f_{11}^I(x_4^*, y_4^*) + q_{11}f_{11}^R(x_4^*, y_4^*) + p_{12}f_{12}^I(x_2^*, y_2^*) \\ \quad + q_{12}f_{12}^R(x_2^*, y_2^*) + p_{13}f_{13}^I(x_3^*, y_3^*) + q_{13}f_{13}^R(x_3^*, y_3^*) = 0, \\ -a_2x_2^* + b_2y_2^* + p_{21}f_{21}^R(x_1^*, y_1^*) - q_{21}f_{21}^I(x_1^*, y_1^*) + p_{22}f_{22}^R(x_5^*, y_5^*) \\ \quad - q_{22}f_{22}^I(x_5^*, y_5^*) + p_{23}f_{23}^R(x_3^*, y_3^*) - q_{23}f_{23}^I(x_3^*, y_3^*) = 0, \\ -a_2y_2^* - b_2x_2^* + p_{21}f_{21}^I(x_1^*, y_1^*) + q_{21}f_{21}^R(x_1^*, y_1^*) + p_{22}f_{22}^I(x_5^*, y_5^*) \\ \quad + q_{22}f_{22}^R(x_5^*, y_5^*) + p_{23}f_{23}^I(x_3^*, y_3^*) + q_{23}f_{23}^R(x_3^*, y_3^*) = 0, \\ -a_3x_3^* + b_3y_3^* + p_{31}f_{31}^R(x_1^*, y_1^*) - q_{31}f_{31}^I(x_1^*, y_1^*) + p_{32}f_{32}^R(x_2^*, y_2^*) \\ \quad - q_{32}f_{32}^I(x_2^*, y_2^*) + p_{33}f_{33}^R(x_6^*, y_6^*) - q_{33}f_{33}^I(x_6^*, y_6^*) = 0, \\ -a_3y_3^* - b_3x_3^* + p_{31}f_{31}^I(x_1^*, y_1^*) + q_{31}f_{31}^R(x_1^*, y_1^*) + p_{32}f_{32}^I(x_2^*, y_2^*) \\ \quad + q_{32}f_{32}^R(x_2^*, y_2^*) + p_{33}f_{33}^I(x_6^*, y_6^*) + q_{33}f_{33}^R(x_6^*, y_6^*) = 0, \\ -\alpha x_4^* + \alpha x_1^* = 0, \\ -\alpha y_4^* + \alpha y_1^* = 0, \\ -\alpha x_5^* + \alpha x_2^* = 0, \\ -\alpha y_5^* + \alpha y_2^* = 0, \\ -\alpha x_6^* + \alpha x_3^* = 0, \\ -\alpha y_6^* + \alpha y_3^* = 0. \end{array} \right. \quad (10)$$

Obviously, zero is an equilibrium point since $f_{jk}^R(0, 0) = 0, f_{jk}^I(0, 0) = 0 (j, k = 1, 2, 3)$. From $-\alpha x_{j+3}^* + \alpha x_j^* = 0, -\alpha y_{j+3}^* + \alpha y_j^* = 0 (j = 1, 2, 3)$ we get $x_{j+3}^* = x_j^*, y_{j+3}^* = y_j^* (j = 1, 2, 3)$. Therefore, system

(10) changes to the following:

$$\left\{ \begin{array}{l} p_{11}f_{11}^R(x_1^*, y_1^*) - q_{11}f_{11}^I(x_1^*, y_1^*) + p_{12}f_{12}^R(x_2^*, y_2^*) - q_{12}f_{12}^I(x_2^*, y_2^*) \\ \quad + p_{13}f_{13}^R(x_3^*, y_3^*) - q_{13}f_{13}^I(x_3^*, y_3^*) = a_1x_1^* - b_1y_1^*, \\ q_{11}f_{11}^R(x_1^*, y_1^*) + p_{11}f_{11}^I(x_1^*, y_1^*) + q_{12}f_{12}^R(x_2^*, y_2^*) + p_{12}f_{12}^I(x_2^*, y_2^*) \\ \quad + q_{13}f_{13}^R(x_3^*, y_3^*) + p_{13}f_{13}^I(x_3^*, y_3^*) = b_1x_1^* + a_1y_1^*, \\ p_{21}f_{21}^R(x_1^*, y_1^*) - q_{21}f_{21}^I(x_1^*, y_1^*) + p_{22}f_{22}^R(x_2^*, y_2^*) - q_{22}f_{22}^I(x_2^*, y_2^*) \\ \quad + p_{23}f_{23}^R(x_3^*, y_3^*) - q_{23}f_{23}^I(x_3^*, y_3^*) = a_2x_2^* - b_2y_2^*, \\ q_{21}f_{21}^R(x_1^*, y_1^*) + p_{21}f_{21}^I(x_1^*, y_1^*) + q_{22}f_{22}^R(x_2^*, y_2^*) + p_{22}f_{22}^I(x_2^*, y_2^*) \\ \quad + q_{23}f_{23}^R(x_3^*, y_3^*) + p_{23}f_{23}^I(x_3^*, y_3^*) = b_2x_2^* + a_2y_2^*, \\ p_{31}f_{31}^R(x_1^*, y_1^*) - q_{31}f_{31}^I(x_1^*, y_1^*) + p_{32}f_{32}^R(x_2^*, y_2^*) - q_{32}f_{32}^I(x_2^*, y_2^*) \\ \quad + p_{33}f_{33}^R(x_3^*, y_3^*) - q_{33}f_{33}^I(x_3^*, y_3^*) = a_3x_3^* - b_3y_3^*, \\ q_{31}f_{31}^R(x_1^*, y_1^*) + p_{31}f_{31}^I(x_1^*, y_1^*) + q_{32}f_{32}^R(x_2^*, y_2^*) + p_{32}f_{32}^I(x_2^*, y_2^*) \\ \quad + q_{33}f_{33}^R(x_3^*, y_3^*) + p_{33}f_{33}^I(x_3^*, y_3^*) = b_3x_3^* + a_3y_3^*. \end{array} \right. \quad (11)$$

System (11) can be written as a matrix form, the matrix of the left hand is C . Obviously, when $x_j^* > 0, y_j^* > 0 (j = 1, 2, 3)$ we have $b_1x_1^* + a_1y_1^* > 0$, and $f_{jk}^R(x^*, y^*) > 0, f_{jk}^I(x^*, y^*) > 0$. However, the left hand can not guarantee greater than zero since C is not a positive definite matrix. The same as when $x_j^* < 0, y_j^* < 0 (j = 1, 2, 3)$ we have $b_1x_1^* + a_1y_1^* < 0$, and $f_{jk}^R(x^*, y^*) < 0, f_{jk}^I(x^*, y^*) < 0$. But the value of the left hand can not guarantee less than zero. This means that system (7) has a unique zero equilibrium.

Lemma 2 Assume that $f_{jk}^R(x, y), f_{jk}^I(x, y) (j, k = 1, 2, 3)$ are continuous bounded functions, $a_j > 0, b_j > 0 (j = 1, 2, \dots, 6)$. Then all solutions of system (7) are uniformly bounded.

Proof Since $f_{jk}^R(x, y), f_{jk}^I(x, y) (j, k = 1, 2, 3)$ are continuous bounded functions, then from the definition $z_4(t) = \int_{-\infty}^t F(t-s)z_1(s)ds$, and the boundedness of $f_{11}^R(x_4(t), y_4(t))$, and $f_{11}^I(x_4(t), y_4(t))$, we get that $x_4(t)$, and $y_4(t)$ are bounded. Similarly, from $z_5(t) = \int_{-\infty}^t F(t-s)z_2(s)ds$, $z_6(t) = \int_{-\infty}^t F(t-s)z_3(s)ds$, and the boundedness of $f_{22}^R(x_5(t), y_5(t))$, $f_{22}^I(x_5(t), y_5(t))$, and $f_{33}^R(x_6(t), y_6(t))$, $f_{33}^I(x_6(t), y_6(t))$ we get that $x_5(t)$, $y_5(t)$, and $x_6(t)$, $y_6(t)$ are bounded. From the equations

$$\left\{ \begin{array}{l} x_4'(t) = -\alpha x_4(t) + \alpha x_1(t), \\ y_4'(t) = -\alpha y_4(t) + \alpha y_1(t), \\ x_5'(t) = -\alpha x_5(t) + \alpha x_2(t), \\ y_5'(t) = -\alpha y_5(t) + \alpha y_2(t), \\ x_6'(t) = -\alpha x_6(t) + \alpha x_3(t), \\ y_6'(t) = -\alpha y_6(t) + \alpha y_3(t). \end{array} \right. \quad (12)$$

we get $x_j(t)$, $y_j(t)$ are bounded ($j = 1, 2, 3$).

3 Main Results

In order to discuss the instability of the trivial solution of system (7) we only need to discuss the instability of the trivial solution of system (8).

Theorem 1 Assume that Lamma1 and Lemma 2 hold. For selected parameters values of a_j, b_j, p_{jk} and q_{jk} ($1 \leq j, k \leq 3$). Let the eigenvalues of matrix $A + B = D$ be δ_i ($i = 1, 2, \dots, 12$). If there exists at least one eigenvalue $\delta_k, k \in \{1, 2, \dots, 12\}$ such that $\delta_k > 0$ or $Re(\delta_k) > 0$, then the unique equilibrium point of system (8) is unstable, implying that system (7) generates an oscillatory solution.

Proof Obviously, if the trivial solution of system (8) is unstable, then the trivial solution of system (7) is also unstable. Therefore, we only need to consider the instability of the trivial solution of system (8). When $\tau = 0$, system (8) reduced to

$$U'(t) = AU(t) + BU(t) = DU(t) \quad (13)$$

The characteristic equation corresponding to system (13) is

$$\det[\lambda I_{ij} - D] = 0. \quad (14)$$

where I_{ij} is the 12 by 12 identity matrix. Since δ_i ($i = 1, 2, \dots, 12$) are eigenvalues of matrix D , therefore we have from (14)

$$\prod_{j=1}^{12} (\lambda - \delta_j) = 0. \quad (15)$$

Since there exists some $\delta_k > 0$ or $Re(\delta_k) > 0$, this means that system (14) has a positive real eigenvalue or an eigenvalue which has a positive real part. Therefore the trivial solution of system (13) is unstable according to the basic result of differential equation. Now we prove that the trivial solution of system (8) also is unstable. Suppose that $U(t)$ is a trivial solution of system (13). It is known that $U(t) \sim U(t - \tau)$ when t is sufficiently large. According to the definition of instability of the trivial solution, for any $\varepsilon > 0$ one can find a sequence $\{t_1, t_2, \dots, t_n, \dots\}$, where $t_1(> \tau)$ is sufficiently large such that $|U(t_k)| > \varepsilon$. When t_1 is sufficiently large, we have $U(t_k) \sim U(t_k - \tau)$, means that $U(t_k)$ also is the solution of system (8). Thus, the trivial solution of system (8) is unstable, implying that the trivial solution of system (7) is unstable. Since system (7) has a unique equilibrium point and all solutions are bounded, this instability of the trivial solution will force system (7) to generate a limit cycle, namely, a periodic oscillation [24], and the appendix of [25].

Set μ $\sum_{j=1}^n \|\mu_j\|$, $\|B\| = \max_{j=1,2,3} \sum_{k=1}^3$

Theorem 2 Assume that Lamma1 and Lemma 2 hold. For selected parameters values of a_j, b_j, p_{jk} and $q_{jk}(1 \leq j, k \leq 3)$. If

$$\mu(A) + \|B\| > 0. \quad (16)$$

Then the unique equilibrium point of system (8) is unstable, implying that system (7) generates an oscillatory solution.

Proof Similar to Theorem 1, we show that the trivial solution of system (8) is unstable, then the trivial solution of system (7) also is unstable. The characteristic equation corresponding to system (8) is

$$\lambda = \mu(A) + \|B\|e^{-\lambda\tau}. \quad (17)$$

or

$$\lambda - \mu(A) - \|B\|e^{-\lambda\tau} = 0. \quad (18)$$

Equation (18) is a transcendental equation. We prove that there is a positive characteristic root of system (18). Let $f(\lambda) = \lambda - \mu(A) - \|B\|e^{-\lambda\tau}$, then $f(\lambda)$ is a continuous function of λ . $f(0) = -\mu(A) - \|B\| = -(\mu(A) + \|B\|) < 0$. Noting that $\lim_{\lambda \rightarrow \infty} e^{-\lambda\tau} = 0$. So, there exists a $\lambda_0 > 0$ such that $f(\lambda_0) = \lambda_0 - \mu(A) - \|B\|e^{-\lambda_0\tau} > 0$. According to the Intermediate Value Theorem, there exists a $\lambda_1 \in (0, \lambda_0)$ such that $f(\lambda_1) = \lambda_1 - \mu(A) - \|B\|e^{-\lambda_1\tau} = 0$. This means that equation (18) has a positive characteristic root. Therefore, the trivial solution of system (8) is unstable, implying that the trivial solution of system (7) is unstable. This instability of the trivial solution will force system (7) to generate a limit cycle, namely, a periodic oscillatory solution.

4 Simulation Results

This simulation is based on system (7). We first select the parameters as $\alpha = 0.24$, time delay $\tau = 0.8$, $a_1 = 0.0065, a_2 = 0.0055, a_3 = 0.0075; b_1 = 0.095, b_2 = 0.085, b_3 = 0.075$; the other parameter values as $p_{11} = 0.5, p_{12} = 0.4, p_{13} = 0.6, p_{21} = -0.075, p_{22} = 0.086, p_{23} = -0.065, p_{31} = 0.55, p_{32} = 0.45, p_{33} = 0.65; q_{11} = 0.95, q_{12} = -0.92, q_{13} = 0.96, q_{21} = 0.25, q_{22} = 0.35, q_{23} = -0.15, q_{31} = 0.15, q_{32} = 0.12, q_{33} = -0.16$. The activation function as $f_{jk}^R(x, y) = f_{jk}^I(x, y) = (\tanh(x) + \tanh(y)) + i(\tanh(x) + \tanh(y))(j, k = 1, 2, 3)$, thus $\frac{\partial f_{jk}^R(0,0)}{\partial x} = \frac{\partial f_{jk}^R(0,0)}{\partial y} = 1$, and $\frac{\partial f_{jk}^I(0,0)}{\partial x} = \frac{\partial f_{jk}^I(0,0)}{\partial y} = 1$ ($j, k = 1, 2, 3$). We

have $c_{14} = d_{14} = p_{11} - q_{11} = -0.4$, $c_{12} = d_{12} = p_{12} - q_{12} = 1.32$, $c_{13} = d_{13} = p_{13} - q_{13} = -0.36$; $m_{14} = n_{14} = p_{11} + q_{11} = 1.45$, $m_{12} = n_{12} = p_{12} + q_{12} = -0.52$, $m_{13} = n_{13} = p_{13} + q_{13} = 1.56$; $c_{25} = d_{25} = p_{22} - q_{22} = -0.264$, $c_{21} = d_{21} = p_{21} - q_{21} = -0.0175$, $c_{23} = d_{23} = p_{23} - q_{23} = 0.085$; $m_{25} = n_{25} = p_{22} + q_{22} = 0.436$, $m_{21} = n_{21} = p_{21} + q_{21} = 0.175$, $m_{23} = n_{23} = p_{23} + q_{23} = -0.215$; $c_{36} = d_{36} = p_{33} - q_{33} = 0.81$, $c_{31} = d_{31} = p_{31} - q_{31} = 0.4$, $c_{32} = d_{32} = p_{32} - q_{32} = 0.33$; $m_{36} = n_{36} = p_{33} + q_{33} = 0.49$, $m_{31} = n_{31} = p_{31} + q_{31} = 0.7$, $m_{32} = n_{32} = p_{32} + q_{32} = 0.57$. Thus, the eigenvalues of matrix C are $1.1537 \pm 1.1291i$, $0.4928 \pm 0.2051i$, $-0.4105 \pm 0.2161i$. This means that C is not a positive definite matrix. The eigenvalues of matrix $A + B$ are $1.3757, 0.0208, -1.3757, -0.4354, -0.2400, -0.2306, 0.0818 \pm 0.1350i, -0.2114 \pm 0.0760i, -0.2400 \pm 0.0001i$. Obviously, 1.3757 is a positive eigenvalue. Based on Theorem 1, there exists a periodic oscillatory solution (see Fig. 1). In order to see the effect of the parameter α , we change $\alpha = 0.86$, the other parameters are the same as in figure 1, we see the oscillatory frequency and amplitude are almost the same as in figure 1 (see Fig. 2). Then we change the activation function as $f_{jk}^R(x, y) = f_{jk}^I(x, y) = (\arctan(x) + \arctan(y)) + i(\arctan(x) + \arctan(y))$ ($j, k = 1, 2, 3$), thus we still have $\frac{\partial f_{jk}^R(0,0)}{\partial x} = \frac{\partial f_{jk}^R(0,0)}{\partial y} = 1$, and $\frac{\partial f_{jk}^I(0,0)}{\partial x} = \frac{\partial f_{jk}^I(0,0)}{\partial y} = 1$ ($j, k = 1, 2, 3$). The parameters are the same as in figure 2, we see the oscillatory frequency almost the same as in figure 2. However, the oscillatory amplitude changes too much (see Fig. 3). Now we select the parameters as $\alpha = 1.6$, time delay $\tau = 0.5$, $a_1 = 0.15, a_2 = 0.25, a_3 = 0.18$; $b_1 = 1.15, b_2 = 1.05, b_3 = 1.18$; $p_{11} = 0.72, p_{12} = 0.25, p_{13} = 0.48, p_{21} = 0.175, p_{22} = -0.16, p_{23} = 0.65, p_{31} = 0.15, p_{32} = -0.24, p_{33} = 0.36$; $q_{11} = -0.25, q_{12} = 0.24, q_{13} = -0.36, q_{21} = 0.18, q_{22} = 0.15, q_{23} = -0.35, q_{31} = 0.28, q_{32} = -0.25, q_{33} = 0.4$. We have $c_{14} = d_{14} = p_{11} - q_{11} = 0.97$, $c_{12} = d_{12} = p_{12} - q_{12} = 0.01$, $c_{13} = d_{13} = p_{13} - q_{13} = 0.84$; $m_{14} = n_{14} = p_{11} + q_{11} = 0.47$, $m_{12} = n_{12} = p_{12} + q_{12} = 0.49$, $m_{13} = n_{13} = p_{13} + q_{13} = 0.12$; $c_{25} = d_{25} = p_{22} - q_{22} = -0.31$, $c_{21} = d_{21} = p_{21} - q_{21} = -0.005$, $c_{23} = d_{23} = p_{23} - q_{23} = 1$; $m_{25} = n_{25} = p_{22} + q_{22} = -0.01$, $m_{21} = n_{21} = p_{21} + q_{21} = 0.355$, $m_{23} = n_{23} = p_{23} + q_{23} = 0.3$; $c_{36} = d_{36} = p_{33} - q_{33} = -0.04$, $c_{31} = d_{31} = p_{31} - q_{31} = -0.13$, $c_{32} = d_{32} = p_{32} - q_{32} = -0.01$; $m_{36} = n_{36} = p_{33} + q_{33} = 0.76$, $m_{31} = n_{31} = p_{31} + q_{31} = 0.43$, $m_{32} = n_{32} = p_{32} + q_{32} = -0.49$. We see that $\|B\| = 1$, $\mu(A) = 1.85$. Therefore, $\|B\| + \mu(A) = 2.85 > 0$. Based on Theorem 2, there exists a periodic oscillatory solution (see Fig. 4).

5 Conclusion

In this paper, we have discussed the oscillatory behavior of the solutions for a complex-valued neural network model with discrete and distributed delays. By means of the mathematical analysis method, two criteria to guarantee the existence of periodic oscillations, which is easy to check, as compared

to find the regions of bifurcation have been proposed. In this network, we decomposed the activation functions and connection weights into their real and imaginary parts, so as to discuss an equivalent real-valued system. The activation function affects the oscillatory amplitude. When these time delay systems generate a periodic oscillatory solution, the delays affect oscillatory frequencies.

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Fig. 1 Oscillation of the solutions. delay: 0.8, alpha=0.24.

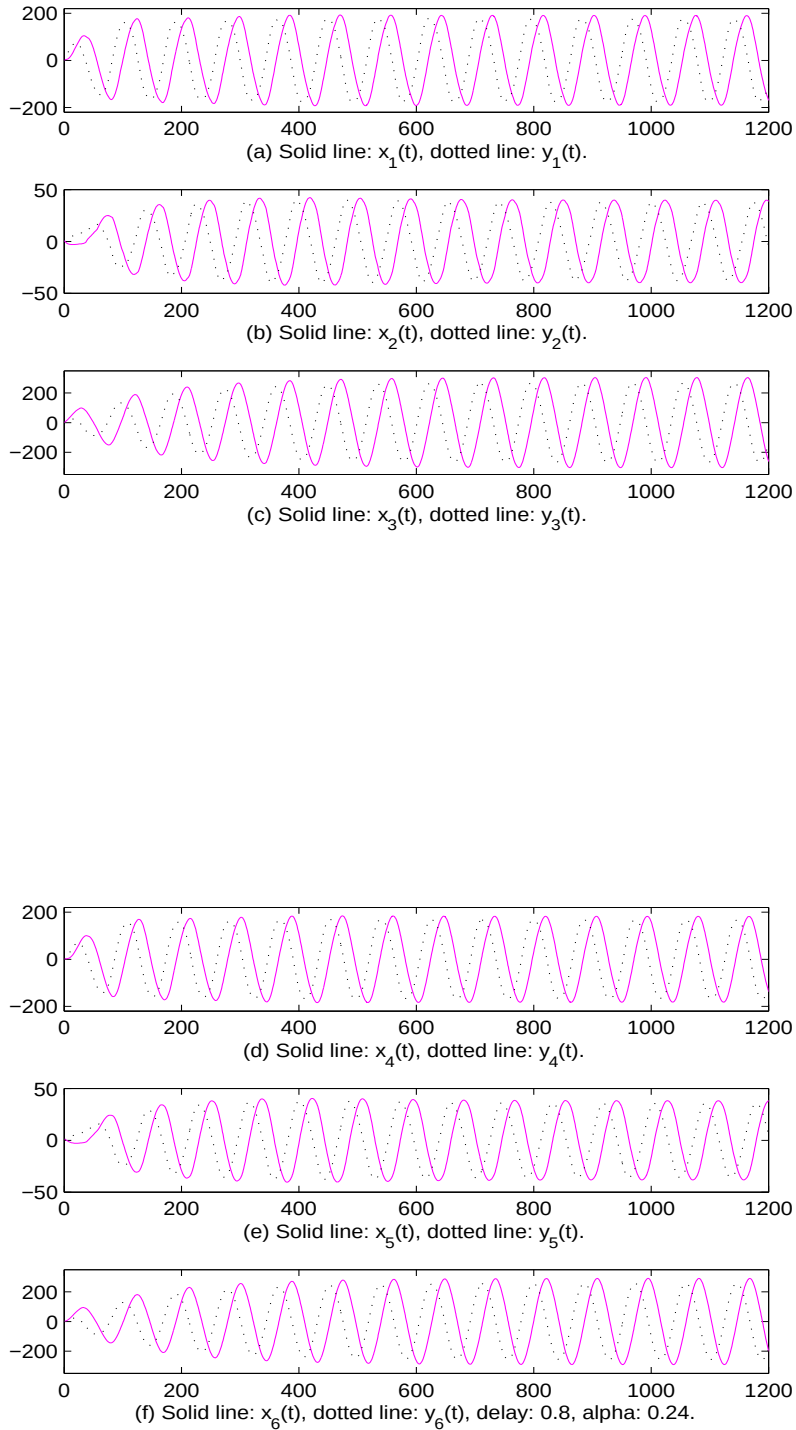
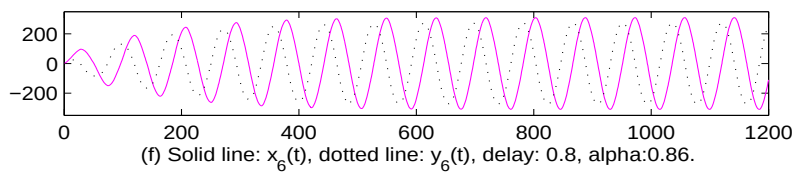
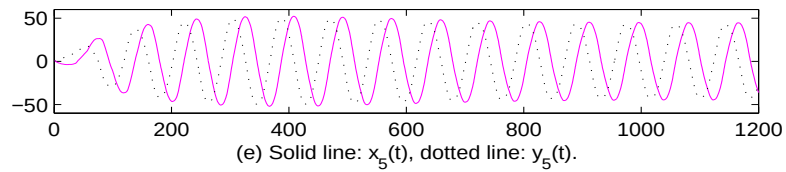
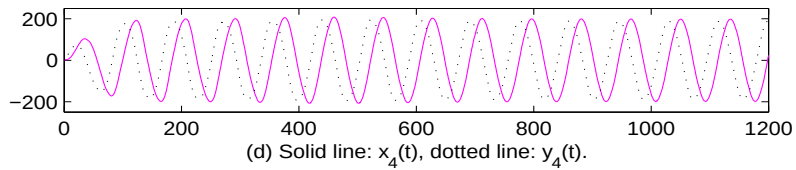
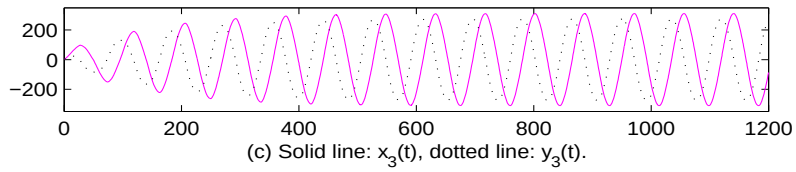
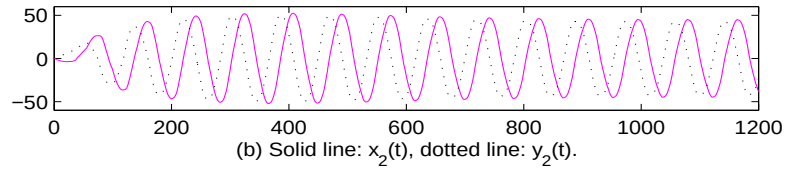
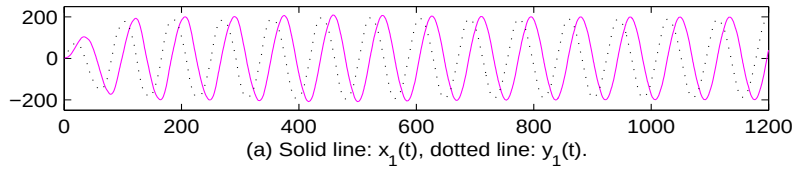


Fig. 2 Oscillation of the solutions, delay: 0.8, $\alpha=0.86$.



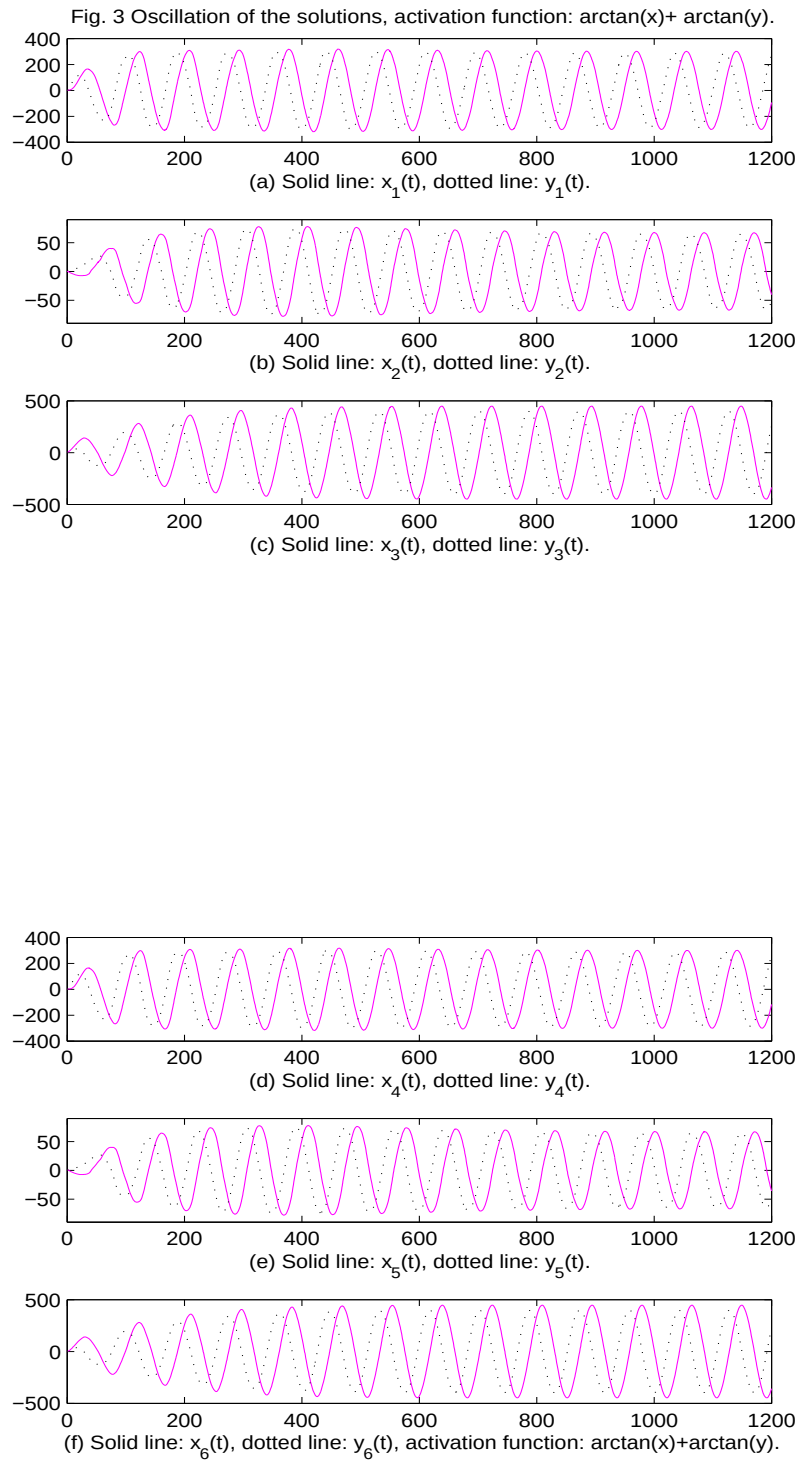


Fig. 4 Oscillation of the solutions, delay: 0.5, alpha: 1.6.

