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# Contrast with the Hille-Yosida's Theorem and the Strongly Continuous Semigroup of Contraction for a Third-order Differential Operator

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**Abstract:** In this work, we prove that the closed right half-plane is contained in the resolvent set of odd-order differential operator A and that the norm of the resolvent operator of A on z with positive real part is bounded by the inverse of the real part of z, which connects us to the Hille-Yosida's Theorem. Furthermore, we explore the connection between being a strongly continuous semigroup of contraction and the dissipativeness of its infinitesimal generator. Finally, we generalize the results obtained for odd-order differential operators.

**Keywords:** Semigroup of contraction, Hille-Yosida's Theorem, third-order differential operator, dissipative operator, Periodic Sobolev spaces, Fourier Theory.

## INTRODUCTION

Let´s remember that $H^s_{per}$ denotes the periodic Sobolev space with s a real number, we consider $a > 0$ and $A := -\partial_x^3 - aI$ is the infinitesimal generator of the semigroup $\{S(t)\}_{t\geq 0}$ in $H^s_{per}$ that we proved in [2]. We are interested in knowing the resolvent set of $A$, and bounding the norm of the resolvent operator of $A$ in the sense of obtaining

$$\|(\lambda I - A)^{-1}\| \leq \frac{1}{Re\,\lambda}, \forall \lambda \in \mathbb{C}_*$$

where $\mathbb{C}_* := \{\lambda \in \mathbb{C}: Re\,\lambda \geq 0\} \subset \rho(A)$, which connects us to the Hille-Yosida's Theorem.

In this analysis we obtain much more, we will obtain that the closed right half-plane is contained in the resolvent set of $A$. And we will generalize the results obtained for an odd-order differential operator.

Furthermore, we will explore the connection between being a strongly continuous semigroup of contraction and the dissipativeness of its infinitesimal generator; we will do this study for A and its generalization $\mathcal{A} := -\partial_x^n - aI$ . So, we also obtain results for the case a=0 and their generalizations.

We can cite [2] and [3], where we find some results related to operator $A$. And we cite [5] and [6] for being a source of inspiration for this work.

The structure of our article is as follows. In section 2, we outline the methodology used and provide the citations for the references consulted. In section 3, we study the resolvent operator of A and its generalization $\mathcal{A}$. In section 4, we study the equivalence between $C_0$-Semigroup of contraction and dissipative operator, and its generalization. Finally, in section 5, we present the conclusions of our study.

## METHODOLOGY

In this article, we mainly employ [2] and [3] as the theoretical framework. In addition, we use the references [4], [1] and [7] for the Fourier theory in $H^s_{per}$, and differential calculus in Banach spaces.

Motivated by proving that A is the infinitesimal generator of a contraction semigroup, obtaining important properties of the resolvent of A, we will compare the results obtained with the famous Hille-Yosida's theorem and its Corollary, from [6] .

Moreover, we will explore the connection between being a strongly continuous semigroup of contraction and the dissipativeness of its infinitesimal generator.

## RESOLVENT OPERATOR OF A AND $\mathcal{A}$ ON $\mathbb{C}_*$

### Resolvent Operator of a

We introduce the following set

$$\mathbb{C}_* := \{\lambda \in \mathbb{C}, Re\,\lambda > 0\}$$

and we denote $\lambda = Re\,\lambda + i\,Im\,\lambda$. Thus, we will prove that $\mathbb{C}_* \subset \rho(A)$.

In Theorem 3.1 from [3] we proved that A is dissipative. That is, $Re < Av, v >_s = -a\,\|v\|_s^2 \leq 0, \forall v \in D(A)$.

So, we will prove that $A$ dissipative implies (3.1).

**Theorem 3.1:** Let $s \in R$ and $a > 0$. Then the operator $A := -\partial_x^3 - aI$ satisfies

$$\|(\lambda I - A)u\|_s \geq Re\,\lambda\,\|u\|_s, \forall u \in D(A), \lambda \in \mathbb{C}_*. \quad (3.1)$$

Proof.- Indeed, let $u \in D(A)$, $u \neq 0$ and $\lambda \in \mathbb{C}_*$ ,

$$\|(\lambda I - A)u\|_s \|u\|_s \geq |< (\lambda I - A)u, u >_s| \geq Re\{< (\lambda I - A)u, u >_s\}. \quad (3.2)$$

Now,

$$< (\lambda I - A)u, u >_s = \lambda \|u\|_s^2 - < Au, u >_s .$$

Then

$$Re < (\lambda I - A)u, u >_s = Re\,\lambda \|u\|_s^2 - Re < Au, u >_s \geq Re\,\lambda\,\|u\|_s^2 \quad (3.3)$$

since $Re < Au, u >_s \leq 0$ .

Using (3.3) in (3.2) we obtain

$$\|(\lambda I - A)u\|_s \|u\|_s \geq Re\,\lambda\,\|u\|_s^2 .$$

Since $u \neq 0$ implies $\|u\|_s > 0$, then

$$\|(\lambda I - A)u\|_s \geq Re\,\lambda\,\|u\|_s .$$

If $u = 0$, equality in (3.1) holds.

■

**Remark 3.1:** From Theorem 3.1 we have that $\lambda I - A$ is injective for all $\lambda \in \mathbb{C}_*$ .

**Theorem 3.2:** Let $s \in R$ and $a > 0$. Then the operator $A := -\partial_x^3 - aI$ satisfies $Im(\lambda I - A) = H_{per}^s, \forall \lambda \in \mathbb{C}_*$.

**Proof:** Indeed, let $\lambda \in \mathbb{C}_*$, $f \in H_{per}^s$, we will prove that there exists $u \in D(A)$ such that $(\lambda I - A)u = f$.

First, we will obtain the candidate for the solution. To achieve this, we apply the Fourier transform to

$$f = \partial_x^3 u + (\lambda + a)u \tag{3.4}$$

with $u \in D(A)$ and obtain

$$\hat{f}(k) = (ik)^3 \hat{u}(k) + (\lambda + a)\hat{u}(k) = (\lambda + a - ik^3)\hat{u}(k). \tag{3.5}$$

We have

$$\lambda + a - ik^3 \neq 0, \forall k \in \mathbb{Z} \tag{3.6}$$

since $|\lambda + a - ik^3| = \sqrt{(Re\,\lambda + a)^2 + (Im\,\lambda - k^3)^2} > 0, \forall k \in \mathbb{Z}$, where $\lambda = Re\,\lambda + i\,Im\,\lambda$.

From (3.5) and (3.6) we obtain

$$\hat{u}(k) = \frac{\hat{f}(k)}{\lambda + a - ik^3}, \forall k \in \mathbb{Z}. \tag{3.7}$$

From which we get our candidate for the solution of (3.4)

$$u = \left[\left(\frac{\hat{f}(k)}{\lambda + a - ik^3}\right)_{k\in\mathbb{Z}}\right]^{\vee}. \tag{3.8}$$

Second, we will prove

$$u \in H_{per}^{s+3} \text{ and } \exists \mathcal{M} > 0, \|u\|_s \leq \|u\|_{s+3} \leq \sqrt{\mathcal{M}}\, \| f \|_s. \tag{3.9}$$

Indeed, from (3.8) we have

$$\begin{aligned} \|u\|_{H_{per}^{s+3}}^2 &= 2\pi \sum_{k=-\infty}^{+\infty} (1 + k^2)^{s+3} . |\hat{u}(k)|^2 \\ &= 2\pi \sum_{k=-\infty}^{+\infty} (1 + k^2)^{s+3} \left|\frac{\hat{f}(k)}{\lambda + a - ik^3}\right|^2 \\ &= 2\pi \sum_{k=-\infty}^{+\infty} (1 + k^2)^s \left|\hat{f}(k)\right|^2 \underbrace{\frac{(1 + k^2)^3}{(Re\,\lambda + a)^2 + (Im\,\lambda - k^3)^2}}_{\mathfrak{S}(k):=} \end{aligned} \tag{3.10}$$

where $\lambda = Re\,\lambda + i\,Im\,\lambda$.

We have that there exists $\mathcal{M} > 0$ such that

$$0 < \mathfrak{S}(k) \leq \mathcal{M}, \forall k \in \mathbb{Z}. \tag{3.11}$$

Using (3.11) in (3.10) we get

$$\|u\|_{s+3}^2 \leq \mathcal{M} . 2\pi \sum_{k=-\infty}^{+\infty} (1 + k^2)^s \left|\hat{f}(k)\right|^2 = \mathcal{M}\|f\|_s^2.$$

As $H_{per}^{s+3} \subset H_{per}^{s}$ and with continuous immersion we get (3.9).

Finally, we will prove $(\lambda I - A)u = f$ in $H_{per}^{s}$ . Indeed,

$$\|(\lambda I - A)u - f\|_s^2 = 2\pi \sum_{k=-\infty}^{+\infty} (1+k^2)^s . \left| (\lambda + a - ik^3)\hat{u}(k) - \hat{f}(k) \right|^2$$

$$= 2\pi \sum_{k=-\infty}^{+\infty} (1+k^2)^s . \left| (\lambda + a - ik^3) \cdot \frac{\hat{f}(k)}{\lambda + a - ik^3} - \hat{f}(k) \right|^2$$

$$= 2\pi \sum_{k=-\infty}^{+\infty} (1+k^2)^s \left| \hat{f}(k) - \hat{f}(k) \right|^2 = 0 . \qquad (3.12)$$

That is, $(\lambda I - A)u - f = 0$.

■

**Theorem 3.3**: Let $s \in R$ and $a > 0$. Then $(\lambda I - A)^{-1}: H_{per}^{s} \to D(A)$ is a bounded linear operator for each $\lambda \in \mathbb{C}_*$ and it satisfies: $\exists \mathcal{M} > 0$ such that

$$\|(\lambda I - A)^{-1}\| \leq \sqrt{\mathcal{M}} , \lambda \in \mathbb{C}_* .$$

**Proof:** The inequality (3.1) implies $\lambda I - A$ injective for all $\lambda \in \mathbb{C}_*$. Thus, from Theorem 3.2, we have that $\lambda I - A$ is bijective then there exists $(\lambda I - A)^{-1}: H_{per}^{s} \to D(A)$ which is linear with $(\lambda I - A)^{-1} f = u$ and from (3.9) it satisfies

$$\|(\lambda I - A)^{-1} f\|_s \leq \|(\lambda I - A)^{-1} f\|_{s+3} \leq \sqrt{\mathcal{M}} \|f\|_s , \forall f \in H_{per}^{s} , \forall \lambda \in \mathbb{C}_* . \qquad (3.13)$$

Then $(\lambda I - A)^{-1}$ is a bouded operator such that $\|(\lambda I - A)^{-1}\| \leq \sqrt{\mathcal{M}}$ holds for all $\lambda \in \mathbb{C}_*$.

■

**Remark 3.2:** Also from Theorems 3.1 and 3.2 we obtain $\|(\lambda I - A)^{-1} f\|_s \leq \frac{1}{Re\,\lambda} \|f\|_s , \forall f \in H_{per}^{s} , \forall \lambda \in \mathbb{C}_*$. That is, $\|(\lambda I - A)^{-1}\| \leq \frac{1}{Re\,\lambda} , \forall \lambda \in \mathbb{C}_*$ .

Therefore, we obtain the following result.

**Corollary 3.1:** Based on the hypothesis of the previous Theorem, we get $\mathbb{C}_* \subset \boldsymbol{\rho(A)}$ and

$$\|(\lambda I - A)^{-1}\| \leq \frac{1}{Re\,\lambda} , \forall \lambda \in \mathbb{C}_*.$$

**Proof:** From Remark 3.2 or Theorems 3.2 and 3.3 we obtain the result.

■

**Corollary 3.2:** Since $\mathbb{R}^+ \subset \mathbb{C}_*$ then $\mathbb{R}^+ \subset \rho(A)$ and $\|(\lambda I - A)^{-1}\| \leq \frac{1}{\lambda} , \forall \lambda > 0$.

**Proof:** From Corollary 3.1 we obtain the result.

■

**Remark 3.3:** Based on the hypothesis of the previous Theorem, using Corollary 3.1 with Theorem 3.10 or Corollary 3.4 from [3], we have $i\mathbb{R} \cup \mathbb{C}_* = \{\lambda \in \mathbb{C} : Re\,\lambda \geq 0\} \subset \rho(A)$ and

$$\|(\lambda I - A)^{-1}\| \leq \begin{cases} \dfrac{1}{Re\,\lambda}, & \lambda \in \mathbb{C}_* \\ \dfrac{1}{a}, & \lambda \in i\mathbb{R}. \end{cases}$$

That is, the closed right half-plane is contained in the resolvent set of A.

**Resolvent Operator of $\mathcal{A}$**

In Theorem 3.13 from [3] we proved that $\mathcal{A}$ is dissipative. That is, $Re < \mathcal{A}v, v >_s = -a\,\|v\|_s^2 \leq 0, \forall v \in D(\mathcal{A})$.

So, we will prove that $\mathcal{A}$ dissipative implies (3.14).

**Theorem 3.4**: Let $s \in R$ , $a > 0$ and n is an odd number such that $n-1$ is not multiple of four. Then the operator $\mathcal{A} := -\partial_x^n - aI$ satisfies

$$\|(\lambda I - \mathcal{A})u\|_s \geq Re\,\lambda \|u\|_s, \forall u \in D(\mathcal{A}), \forall \lambda \in \mathbb{C}_*. \quad (3.14)$$

**Proof:** Its proof is analogous to the demonstration of Theorem 3.1.

■

**Remark 3.4**: From Theorem 3.4 we have that $\lambda I - \mathcal{A}$ is injective for all $\boldsymbol{\lambda} \in \mathbb{C}_*$ .

**Theorem 3.5:** Let $s \in R$ , $a > 0$ and n is an odd number such that $n-1$ is not multiple of four. Then the operator $\mathcal{A} := -\partial_x^n - aI$ satisfies $Im(\lambda I - \mathcal{A}) = H_{per}^s, \forall \lambda \in \mathbb{C}_*$.

**Proof:** Its proof is similar to the demonstration of Theorem 3.2.

■

**Theorem 3.6**: Let $s \in R$ , $a > 0$ and n is an odd number such that $n-1$ is not multiple of four. Then $(\lambda I - \mathcal{A})^{-1}: H_{per}^s \to D(\mathcal{A})$ is a bounded linear operator for each $\lambda \in \mathbb{C}_*$ and it satisfies: $\exists \mathbb{M} > 0$ ,

$$\|(\lambda I - \mathcal{A})^{-1}\| \leq \sqrt{\mathbb{M}}, \lambda \in \mathbb{C}_* .$$

**Proof:** Its proof is analogous to the demonstration of Theorem 3.3.

■

**Remark 3.5:** Also from Theorems 3.4 and 3.5 we obtain $\|(\lambda I - \mathcal{A})^{-1} f\|_s \leq \frac{1}{Re\,\lambda}\|f\|_s, \forall f \in H_{per}^s, \forall \lambda \in \mathbb{C}_*$. That is, $\|(\lambda I - \mathcal{A})^{-1}\| \leq \frac{1}{Re\,\lambda}$ , $\forall \lambda \in \mathbb{C}_*$ . Therefore, we obtain the following result.

**Corollary 3.3:** Based on the hypothesis of the previous Theorem, we obtain $\mathbb{C}_* \subset \rho(\mathcal{A})$ and

$$\|(\lambda I - \mathcal{A})^{-1}\| \leq \frac{1}{Re\,\lambda}\,, \forall \lambda \in \mathbb{C}_*\,.$$

**Proof:** From Remark 3.5 or Theorems 3.5 and 3.6 we obtain the result.

■

**Corollary 3.4:** Since $\mathbb{R}^+ \subset \mathbb{C}_*$ then $\mathbb{R}^+ \subset \rho(\mathcal{A})$ and $\|(\lambda I - \mathcal{A})^{-1}\| \leq \frac{1}{\lambda}\,, \forall\, \lambda > 0$.

**Proof:** From Corollary 3.3 we obtain the result.

■

**Remark 3.6:** Based on the hypothesis of the previous Theorem, using Corollary 3.3 with Theorem 3.22 or Corollary 3.8 from [3], we have $i\mathbb{R} \cup \mathbb{C}_* = \{\lambda \in \mathbb{C}: Re\,\lambda \geq 0\} \subset \rho(\mathcal{A})$ and

$$\|(\lambda I - \mathcal{A})^{-1}\| \leq \begin{cases} \dfrac{1}{Re\,\lambda}\,, & \lambda \in \mathbb{C}_* \\ \dfrac{1}{a}\,, & \lambda \in i\mathbb{R}\,. \end{cases}$$

That is, the closed right half-plane is contained in the resolvent set of $\mathcal{A}$.

**Remark 3.7**: Similar results are also obtained when $n$ is an odd number such that $n-1$ is multiple of four, in this case, note that $(ik)^n = ik^n, \forall k \in \mathbb{Z}$.

**Case $a = 0$**

**Theorem 3.7:** Let $s \in R$ and $a = 0$. Then the operator $A := -\partial_x^3 - aI = -\partial_x^3$ is dissipative on $H_{per}^s$ where $D(A) = H_{per}^{s+3}$ . That is,

$$Re\{< Au, u >_s\} = 0, \forall u \in H_{per}^{s+3}. \tag{3.15}$$

Moreover, $< Au, u >_s = i\delta$, $\forall u \in H_{per}^{s+3}$ where

$$\mathbb{R} \ni \delta := 2\pi \sum_{k=-\infty}^{+\infty} (1+k^2)^s k^3 |\hat{u}(k)|^2$$

Proof.- Let $u \in H_{per}^{s+3}$,

$$\begin{aligned} < Au, u >_s &= < -\partial_x^3 u, u >_s \\ &= \underbrace{i.\,2\pi \sum_{k=-\infty}^{+\infty} (1+k^2)^s k^3 |\hat{u}(k)|^2}_{\delta :=}\,. \end{aligned} \tag{3.16}$$

At this point, we will prove the convergence of the series (3.16). Indeed, using the inequality: $|k|^3 \leq |k|^6 = (|k|^2)^3 \leq (1+|k|^2)^3$ and $u \in H_{per}^{s+3}$ , we have

$$\left| \sum_{k=-\infty}^{+\infty} (1+k^2)^s k^3 |\hat{u}(k)|^2 \right| \leq \sum_{k=-\infty}^{+\infty} (1+k^2)^s |k|^3 |\hat{u}(k)|^2$$

$$\leq \sum_{k=-\infty}^{+\infty} (1+k^2)^s (1+|k|^2)^3 |\hat{u}(k)|^2$$

$$= \sum_{k=-\infty}^{+\infty} (1+k^2)^{s+3} |\hat{u}(k)|^2 = \frac{1}{2\pi} \|u\|_{s+3}^2 < \infty .$$

Then the series (3.16) is convergent, that is,

$$< Au, u >_s = i\delta \text{, with } \delta \in R . \tag{3.17}$$

Finally, from equality (3.17) we obtain $Re\{< Au, u >_s\} = 0$ , for all $u \in H_{per}^{s+3}$.

■

**Theorem 3.8:** Let $s \in R$ , then the operator $A := -\partial_x^3$ satisfies inequality (3.1).

**Proof:** The proof is similar to the demonstration of Theorem 3.1 where the Theorem 3.7 is used.

■

**Remark 3.8:** From Theorem 3.8 we have that $\lambda I - A$ is injective for all $\lambda \in \mathbb{C}_*$.

**Theorem 3.9:** Let $s \in R$ , then the operator $A := -\partial_x^3$ satisfies $Im(\lambda I - A) = H_{per}^s$ , $\forall \lambda \in \mathbb{C}_*$.

**Proof:** The proof is similar to the demonstration of Theorem 3.2.

■

**Theorem 3.10:** Let $s \in R$ and $a = 0$. Then $(\lambda I - A)^{-1}: H_{per}^s \to D(A)$ is a bounded linear operator for each $\lambda \in \mathbb{C}_*$ and it satisfies: $\exists \mathfrak{B} > 0$ such that

$$\|(\lambda I - A)^{-1}\| \leq \sqrt{\mathfrak{B}}, \lambda \in \mathbb{C}_* .$$

**Proof:** The proof is consequence of Theorems 3.8 and 3.9.

■

**Remark 3.9:** Also, from Theorems 3.8 and 3.9 we obtain $\|(\lambda I - A)^{-1} f\|_s \leq \frac{1}{Re\,\lambda} \|f\|_s$ , $\forall f \in H_{per}^s$ , $\forall \lambda \in \mathbb{C}_*$. That is,

$$\|(\lambda I - A)^{-1}\| \leq \frac{1}{Re\,\lambda} , \forall \lambda \in \mathbb{C}_* .$$

Therefore, we obtain the following result.

**Corollary 3.5:** Based on the hypothesis of the previous Theorem, we get $\mathbb{C}_* \subset \rho(A)$ and

$$\|(\lambda I - A)^{-1}\| \leq \frac{1}{Re\,\lambda} , \forall \lambda \in \mathbb{C}_*.$$

That is, the open right half-plane is contained in the resolvent set of A.

**Proof:** From Remark 3.9 or Theorems 3.9 and 3.10 we obtain the result.

■

**Corollary 3.6:** Since $\mathbb{R}^+ \subset \mathbb{C}_*$ then $\mathbb{R}^+ \subset \rho(A)$ and $\|(\lambda I - A)^{-1}\| \leq \frac{1}{\lambda}$ , $\forall\, \lambda > 0$.

**Proof:** From Corollary 3.5 we obtain the result.

■

**Remark 3.10:** If $a = 0$ then $0 \notin \rho(A)$.

**Remark 3.11:** If $a = 0$ then the $\{S(t)\}_{t\geq 0}$ semigroup is not exponentially stable (See [1]).

Next, we will generalize the results obtained.

**Theorem 3.11:** Let $s \in R$ and $a = 0$ and $n$ is an odd number such that $n - 1$ is not multiple of four. Then the operator $\mathcal{A} := -\partial_x^n - aI = -\partial_x^n$ is dissipative on $H_{per}^s$ where $D(\mathcal{A}) = H_{per}^{s+n}$. That is,

$$Re\{< \mathcal{A}u, u >_s\} = 0, \forall u \in H_{per}^{s+n}. \tag{3.18}$$

Moreover, $< \mathcal{A}u, u >_s = i\delta$, $\forall u \in H_{per}^{s+n}$ where

$$\mathbb{R} \ni \delta := 2\pi \sum_{k=-\infty}^{+\infty} (1 + k^2)^s k^n |\hat{u}(k)|^2$$

**Proof:** The proof is similar to the demonstration of Theorem 3.8.

■

**Theorem 3.12:** Let $s \in R$ and n is an odd number such that n-1 is not multiple of four. Then the operator $\mathcal{A} := -\partial_x^n$ satisfies inequality (3.14).

**Proof:** The proof is similar to the demonstration of Theorem 3.1 where the Theorem 3.11 is used.

■

**Remark 3.12**: From Theorem 3.12 we have that $\lambda I - \mathcal{A}$ is injective for all $\lambda \in \mathbb{C}_*$.

**Theorem 3.13:** Let $s \in R$ and n is an odd number such that n-1 is not multiple of four. Then the operator $\mathcal{A} := -\partial_x^n$ satisfies $Im(\lambda I - \mathcal{A}) = H_{per}^s$ , $\forall\, \lambda \in \mathbb{C}_*$.

**Proof:** The proof is similar to the demonstration of Theorem 3.2.

■

**Theorem 3.14:** Let $s \in R$ , $a = 0$ and n is an odd number such that n-1 is not multiple of four. Then $(\lambda I - \mathcal{A})^{-1}: H_{per}^s \to D(\mathcal{A})$ is a bounded linear operator for each $\lambda \in \mathbb{C}_*$ and it satisfies: $\exists \mathfrak{L} > 0$ such that

$$\|(\lambda I - \mathcal{A})^{-1}\| \leq \sqrt{\mathfrak{L}}, \lambda \in \mathbb{C}_* .$$

**Proof:** The proof is consequence of Theorems 3.12 and 3.13.

■

**Remark 3.13**: Also from Theorems 3.12 and 3.13 we obtain $\|(\lambda I-\mathcal{A})^{-1}f\|_s \leq \frac{1}{Re\,\lambda}\|f\|_s, \forall f \in H^s_{per}, \forall\, \lambda \in \mathbb{C}_*$. That is,

$$\|(\lambda I-\mathcal{A})^{-1}\| \leq \frac{1}{Re\,\lambda}\,,\ \forall\, \lambda \in \mathbb{C}_*\ .$$

Therefore, we obtain the following result.

**Corollary 3.9:** Based on the hypothesis of the previous Theorem, we get $\mathbb{C}_* \subset \rho(\mathcal{A})$ and

$$\|(\lambda I-\mathcal{A})^{-1}\| \leq \frac{1}{Re\,\lambda}\,, \forall\, \lambda \in \mathbb{C}_*.$$

That is, the open right half-plane is contained in the resolvent set of $\mathcal{A}$.

**Proof:** From Remark 3.13 or Theorems 3.13 and 3.14 we obtain the result.

■

**Corollary 3.10**: Since $\mathbb{R}^+ \subset \mathbb{C}_*$ then $\mathbb{R}^+ \subset \rho(\mathcal{A})$ and $\|(\lambda I-\mathcal{A})^{-1}\| \leq \frac{1}{\lambda}, \forall\, \lambda > 0$.

**Proof:** From Corollary 3.9 we obtain the result.

■

**Remark 3.14:** If $a = 0$ then $0 \notin \rho(\mathcal{A})$.

**Remark 3.15:** If $a = 0$ then the $\{\mathcal{T}(t)\}_{t\geq 0}$ semigroup is not exponentially stable (See [1]).

**Remark 3.16:** Analogous results are also obtained when $n$ is an odd number such that $n-1$ is multiple of four, in this case, note that $(ik)^n = ik^n, \forall k \in \mathbb{Z}$.

**Comments**

We have already the following statements, where $A = -\partial_x^3 - aI$ is a linear operator. Remember that items 1 and 2 were proved in [2] and item 3 in [3] for $a > 0$. On the other hand, items 1 and 2 were proved in [1] for $a = 0$.

$A$ is closed and $\overline{D(A)} = H^s_{per}$ , where D(A)= $H^{s+3}_{per}$.

A is the infinitesimal generator of a $C_o$ semigroup of contractions $\{S(t)\}_{t\geq 0}$

$\mathbb{R}^+ \subseteq \rho(A)$ and for every $\lambda > 0$

$$\|(\lambda I - A)^{-1}\| \leq \frac{1}{\lambda}$$

$\mathbb{C}_* \subset \rho(A)$ and for every $\lambda \in \mathbb{C}_*$

$$\|(\lambda I - A)^{-1}\| \leq \frac{1}{Re\,\lambda}$$

Now, we will analyze its connection with Hille-Yosida's Theorem 3.1 from [6]. By Hille-Yosida's Theorem we have that item 2 implies items 1 and 3 and that the reciprocal is fulfilled. We proved it without resorting to this Theorem. Note also that using item 4 we get item 3.

Another commonly used result is the Corollary 3.6 from [6], which tells us that item 2 implies item 4. Note that we naturally proved it without resorting to this Corollary.

This analysis also applies to $\mathcal{A} = -\partial_x^n - aI$ when n is odd. Thus, we have compared our results with two important results: Hille-Yosida's Theorem and its Corollary, that motivated this study.

This study is limited to operators of type: $\mathcal{A} = -\partial_x^n - aI$ in $H_{per}^s$ where n is an odd number; analogous to this can be established for n multiple of four in which, for example, an equality is obtained: $Re < \mathcal{A}v, v >_s = -a\,\|v\|_s^2 - \left\|\partial_x^{n/2} v\right\|_s^2 \leq 0, \forall v \in D(\mathcal{A})$ , unlike the odd case in which the equality $Re < \mathcal{A}v, v >_s = -a\,\|v\|_s^2 \leq 0, \forall v \in D(\mathcal{A})$ is obtained.

Let us remember that in the case studied we do obtain the explicit form of the solution, which is a great advantage; whereas Hille-Yosida's Theorem allow us to know theoretically that exists a solution, without knowing the explicit form of the solution (as in the applications in Liu-Zheng [5] or Pazy [6]). Hence the relevance of this important Theorem.

## EQUIVALENCE BETWEEN SEMIGROUP OF CONTRACTION AND DISSIPATIVE OPERATOR

**Theorem 4.1:** Let $H_{per}^s$ be a Hilbert space and the $C_o$- semigroup $\{S(t)\}_{t\geq 0}$ from [2] such that $A$ is its infinitesimal generator, then the following equivalence holds: $\{S(t)\}_{t\geq 0}$ is of contraction if and only if $A$ is dissipative, that is, $Re\ < Av, v >_s = -a\|v\|_s^2 \leq 0, \forall v \in H_{per}^{s+3} = D(A)$.

**Proof:** As $\{S(t)\}_{t\geq 0}$ is of contraction then

$$\|S(t)\| \leq 1, \forall t \geq 0. \tag{4.1}$$

Then, let $u \in H_{per}^s - \{0\}$,

$$\langle S(h)u - u, u\rangle_s = \langle S(h)u, u\rangle_s - \langle u, u\rangle_s = \langle S(h)u, u\rangle_s - \|u\|_s^2 \tag{4.2}$$

Taking part real, we obtain

$$Re\ \langle S(h)u - u, u\rangle_s = Re\ \langle S(h)u, u\rangle_s - \|u\|_s^2\ , \tag{4.3}$$

Using $Re(z) \leq |Re(z)| \leq |z|, \forall z \in \mathbb{C}$ , Hölder's inequality and (4.1), we obtain

$$\begin{aligned} Re\ \langle S(h)u - u, u\rangle_s &\leq |\ \langle S(h)u, u\rangle_s| - \|u\|_s^2 \leq \|S(h)u\|_s \|u\|_s - \|u\|_s^2 \\ &\leq \|u\|_s^2 - \|u\|_s^2 = 0 \end{aligned} \tag{4.4}$$

Therefore,

$$Re\ \langle S(h)u - u, u\rangle_s \leq 0, \forall u \in H_{per}^s. \tag{4.5}$$

In particular,

$$Re\ \langle S(h)v - v, v\rangle_s \leq 0, \forall v \in H^{s+3}_{per} = D(A). \tag{4.6}$$

If $v \in D(A)$ then

$$Re\ \langle \frac{S(h)v-v}{h}, v\rangle_s \leq 0, \forall h > 0. \tag{4.7}$$

As $\frac{S(h)v-v}{h} \to Av$ when $h \to 0^+$ and $\langle \cdot,\cdot \rangle_s$ is continuous then

$$\langle \frac{S(h)v-v}{h}, v\rangle_s \to \langle Av, v\rangle_s \text{ when } h \to 0^+. \tag{4.8}$$

Therefore,

$$\text{Re}\ \langle \frac{S(h)v-v}{h}, v\rangle_s \to Re\ \langle Av, v\rangle_s \text{ when } h \to 0^+. \tag{4.9}$$

$$\text{Im}\ \langle \frac{S(h)v-v}{h}, v\rangle_s \to Im\ \langle Av, v\rangle_s \text{ when } h \to 0^+. \tag{4.10}$$

Using (4.9) in (4.7), we get

$$Re\ \langle Av, v\rangle_s \leq\ 0.$$

Then

$$Re\ \langle Av, v\rangle_s \leq\ 0, \forall v \in D(A). \tag{4.11}$$

Now, we want to know $Re\ \langle Av, v\rangle_s$ explicitly; what is $Re\ \langle Av, v\rangle_s$ ?

Let $h > 0$, we know

$$\langle \frac{S(h)u-u}{h}, u\rangle_s = \frac{2\pi}{h} \sum_{k=-\infty}^{+\infty} (1+k^2)^s \{e^{(ik^3-a)h}\, \hat{u}(k) - \hat{u}(k)\}\overline{\hat{u}(k)}$$

$$= 2\pi \sum_{k=-\infty}^{+\infty} (1+k^2)^s \frac{\{e^{(ik^3-a)h} - 1\}}{h} |\hat{u}(k)|^2$$

for $u \in H^s_{per}$.

In particular,

$$\langle \frac{S(h)v-v}{h}, v\rangle_s = 2\pi \sum_{k=-\infty}^{+\infty} (1+k^2)^s \frac{\{e^{(ik^3-a)h} - 1\}}{h} |\hat{v}(k)|^2 \tag{4.12}$$

for $v \in H^{s+3}_{per} = D(A)$.

Using the Weierstrass M-Test we have that the series (4.12) converges uniformly and consequently it is possible to exchange limits and obtain

$$\lim_{h\to 0^+} \langle \frac{S(h)v-v}{h}, v\rangle_s = 2\pi \sum_{k=-\infty}^{+\infty} (1+k^2)^s \lim_{h\to 0^+} \left\{\frac{e^{(ik^3-a)h} - 1}{h}\right\} |\hat{v}(k)|^2$$

$$= 2\pi \sum_{k=-\infty}^{+\infty} (1+k^2)^s \{ik^3 - a\}\, |\hat{v}(k)|^2$$

$$= i\ 2\pi \sum_{k=-\infty}^{+\infty} (1+k^2)^s k^3\, |\hat{v}(k)|^2 - a\|v\|_s^2 \tag{4.13}$$

As $\langle\cdot,\cdot\rangle_s$ is continuous and $v \in D(A)$, from (4.13) we obtain

$$\langle Av, v\rangle_s = i\,2\pi \sum_{k=-\infty}^{+\infty} (1+k^2)^s k^3\, |\hat{v}(k)|^2 - a\|v\|_s^2 \tag{4.14}$$

Taking the real part of (4.14), we get

$$Re\,\langle Av, v\rangle_s = -a\|v\|_s^2 \tag{4.15}$$

and therefore $Re\,\langle Av, v\rangle_s \le 0$.

Thus, (4.15) is the answer to our question. Furthermore, it is observed that Theorem 3.1 from [3] also answers our question.

Reciprocally, as A is dissipative then

$$Re\,\langle Av, v\rangle_s \le 0, \forall v \in H_{per}^{s+3} = D(A). \tag{4.16}$$

Let $\vartheta \in D(A)$, we define $u(t) := S(t)\vartheta$ , then $u$ satisfies:

$$\begin{vmatrix} u_t = Au \\ u(0) = \vartheta \end{vmatrix}$$

And using (4.16)

$$\frac{1}{2}\frac{\partial}{\partial t}\{\|u\|_s^2\} = \frac{1}{2}\frac{\partial}{\partial t}\{\langle u, u\rangle_s\} = Re\,\langle u_t, u\rangle_s = Re\,\langle Au, u\rangle_s \le 0$$

since $u(t) \in D(A)$. That is, $\frac{\partial}{\partial t}\{\|u(t)\|_s^2\} \le 0$. Thus, $\|u(\cdot)\|_s^2$ is non-increasing. Then $\|u(\cdot)\|_s$ is non-increasing.

So,

$$\|S(t)\vartheta\|_s = \|u(t)\|_s \le \|u(0)\|_s = \|\vartheta\|_s\,, \forall t \ge 0.$$

That is,

$$\|S(t)\vartheta\|_s \le \|\vartheta\|_s\,, \forall \vartheta \in D(A), \forall t \ge 0. \tag{4.17}$$

By density, we obtain

$$\|S(t)\varphi\|_s \le \|\varphi\|_s\,, \forall \varphi \in H_{per}^s, \forall t \ge 0. \tag{4.18}$$

Finally, from (4.18) we get $\|S(t)\| \le 1, \forall t \ge 0$.

■

**Corollary 4.1:** *If* $\{S(t)\}_{t\ge 0}$ *is of contraction then* $Re\,\langle Av, v\rangle_s = -a\|v\|_s^2 \le 0$ , $\forall v \in H_{per}^{s+3} = D(A)$. *Moreover,* $\langle Av, v\rangle_s = i\delta - a\|v\|_s^2$ , $\forall v \in H_{per}^{s+3}$ *where*

$$\mathbb{R} \ni \delta = 2\pi \sum_{k=-\infty}^{+\infty} (1+k^2)^s k^3\, |\hat{v}(k)|^2.$$

**Theorem 4.2:** *Let* $H_{per}^s$ *be a Hilbert space,* $a = 0$ *and the* $C_o$*- semigroup* $\{S(t)\}_{t\ge 0}$ *from [1] such that A is its infinitesimal generator, then the following equivalence holds:* $\|S(t)\| = 1, \forall t \ge 0$ *if and only if* $Re\, < Av, v >_s = 0, \forall v \in H_{per}^{s+3} = D(A)$.

**Proof:** The proof is similar to demonstrate of Theorem 4.1

■

Next, we will generalize the results obtained.

**Theorem 4.3:** *Let $H^s_{per}$ be a Hilbert space, n is an odd number such that $n-1$ is not multiple of four and the $C_o$- semigroup $\{\mathcal{T}(t)\}_{t\geq 0}$ from [2] such that $\mathcal{A}$ is its infinitesimal generator, then the following equivalence holds: $\{\mathcal{T}(t)\}_{t\geq 0}$ is of contraction if and only if $\mathcal{A}$ is dissipative, that is, $Re < \mathcal{A}v, v >_s = -a\|v\|_s^2 \leq 0, \forall v \in H^{s+n}_{per} = D(\mathcal{A})$.*

**Proof:** The proof is analogous to the demonstration of Theorem 4.1.

■

**Corollary 4.2:** *If $\{\mathcal{T}(t)\}_{t\geq 0}$ is of contraction then $Re\,\langle \mathcal{A}v, v\rangle_s = -a\|v\|_s^2 \leq 0$ , $\forall v \in H^{s+n}_{per} = D(\mathcal{A})$.*

*Moreover, $\langle \mathcal{A}v, v\rangle_s = i\delta - a\|v\|_s^2$ , $\forall v \in H^{s+n}_{per}$ where*

$$\mathbb{R} \ni \delta = 2\pi \sum_{k=-\infty}^{+\infty} (1+k^2)^s k^n \, |\hat{v}(k)|^2.$$

**Theorem 4.4:** *Let $H^s_{per}$ be a Hilbert space, $a = 0$, n is an odd number such that $n-1$ is not multiple of four and the $C_o$- semigroup $\{\mathcal{T}(t)\}_{t\geq 0}$ from [1] such that $\mathcal{A}$ is its infinitesimal generator, then the following equivalence holds: $\|\mathcal{T}(t)\| = 1, \forall t \geq 0$ if and only if $Re < \mathcal{A}v, v >_s = 0, \forall v \in H^{s+n}_{per} = D(\mathcal{A})$.*

**Proof:** The proof is similar to demonstrate of Theorem 4.1

■

**Remark 4.1:** Similar results to Theorem 4.3, Corollary 4.2 and Theorem 4.4 are obtained when $n$ is an odd number such that $n-1$ is multiple of four, where

$$\mathbb{R} \ni \delta = -2\pi \sum_{k=-\infty}^{+\infty} (1+k^2)^s k^n \, |\hat{v}(k)|^2.$$

## CONCLUSIONS

Using properties of periodic Sobolev spaces, we proved that the closed right half-plane is contained in the resolvent set of odd-order differential operator $A$ with $\boldsymbol{a} > \mathbf{0}$, and that the norm of the resolvent operator of $A$ on $\lambda$ with positive real part is bounded by the inverse of the real part of $\lambda$ , which connects us to the Hille-Yosida's Theorem. Also, we proved that the open right half-plane is contained in the resolvent set of odd-order differential operator $A$ when a=0. Likewise, it was observed that if one dissipation is eliminated (a = 0), then the semigroup is not exponentially stable. Thus, we have compared our results with two important results: Hille-Yosida's Theorem and its Corollary, that motivated this study. Furthermore, we explored the connection between being a strongly continuous semigroup of contraction and the dissipativeness of its infinitesimal generator for $a \geq 0$. Finally, we generalized the results obtained.

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