ISSN: 2634-9221
European Journal of Applied Sciences; Vol. 14 No. 04 (2026) (177-181)
https://doi.org/10.14738/aivp.1404.12015

Free Fall and the Rotation of Earth

Thanassis E. Dialynas

Physics Department – University of Crete

Abstract: The "free fall" of a heavy object, in a uniform gravitational field from h=10m and no initial velocity is discussed (in the local rotating system of Earth). Especially the deviation to the ''vertical line'' is tried to be calculated which finally depends on what you define as a ''vertical line''.

Keywords: free fall, rotation of Earth, vertical line, perpendicular plumb line.

INTRODUCTION

According to the solution of Newton's equations in a rotational frame (non-inertial frame like this existing in a limited locus on the surface of Earth), a heavy object falling freely (without initial velocity in this system), is expected to deviate from the vertical line (joining the initial point A of the trajectory with the point B on the ground and the point C at the center of the earth) by a deviation of approximately 1,720 cm point B' to the south if it is released from the point (A) h=10m (See figure I).

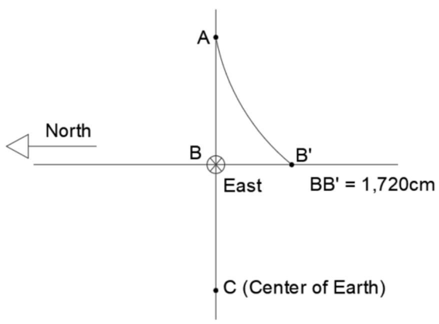


Figure I

One possible way to locate point B is to exert a plumb line from point A down to the ground reaching point "B". Then if you remove the plumb line and leave freely, a heavy body from A, it will fall to the point B' (as we think). If one does so he will be found at a surprise: The distance BB' is approximately 0,6mm and not 1,720cm: What happens? Have we a failure of Newton's equation in a non-inertial system?

The answer is no. What probably happens is that the plumb line is not vertical i.e. does not point to the center of earth, because of the centrifugal forces that are exerted to it, due to the rotation of Earth: So, the plumb line points to a point B" (also to the South) such that BB"=1,714cm.

From which we finally have BB' - BB"=B'B"=0,6mm.

MAIN THEME

Suppose we have a body moving very near to a point S on the surface of earth [where S is considered fixed on the surface of earth]. Then the vector $\vec{r}$ following the body from the center of earth can be written $\vec{r} = \vec{R}_s + \vec{q} = \vec{R} + \vec{q}$ where $\vec{q}$ is the vector connecting the point S with the body (See Figure II).

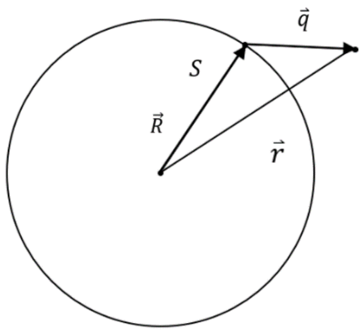


Figure II

$$\vec{r} = \vec{R}_s + \vec{q} = \vec{R} + \vec{q} \tag{1}$$

In addition, we consider $|\vec{q}| << R$ for instance like the motion of a pendulum or a falling body from let's say h≈10m and that the velocity of the body with respect to the point S is $|\vec{U}_q| << \omega R$ where ω is the angular speed of earth and R its radius.

Actually, $U_e = \omega R = 1667{,}9238996 \frac{km}{h} = 463{,}31219433 \frac{m}{sec}$ for R = 6371km, which is the velocity of the equator with respect to the center of the earth.

For further use we also calculate $g_\omega = \omega^2 R$.

$g_\omega = \omega^2 R = 0{,}0336930435 \frac{m}{sec^2}$ which is the centrifugal force of a body near to the equator.

Newton's equation can be written only from an inertial frame or better from a motionless frame of reference. Let us consider the earth's center as motionless. Then we can write:

$$m \frac{d^2 \vec{r}}{dt^2} = m\vec{g} \tag{2}$$

where $\vec{g}$ is the uniform gravitational field and $\vec{g} = -g_o \hat{R} = -g_0 \hat{Z}'$ and $\vec{g} = \hat{Z}'$ is the unit vector that points perpendicularly up at the point S.

Now:

$$\frac{d^2 \vec{r}}{dt^2} = \frac{d^2 \vec{R}}{dt^2} + \frac{d^2 \vec{q}}{dt^2} \tag{3}$$

which gives

$$\frac{d^2 \vec{q}}{dt^2} = \vec{g} - \frac{d^2 \vec{R}}{dt^2} \tag{4}$$

Since $\vec{g}$ is a vector in a rotating frame we must have in general

$$\frac{d^2\vec{q}}{dt} = \ddot{\vec{q}} + \dot{\vec{\omega}}x\vec{q} + 2\left(\vec{\omega}x\dot{\vec{q}}\right) + \vec{\omega}x(\vec{\omega}x\vec{q})[1,2,3] \tag{5}$$

The derivatives $\dot{\vec{q}}$, $\ddot{\vec{q}}$ mean the change of position inside the local non-inertial frame of reference and the external products with $\vec{\omega}$ are related to the acceleration that the moving bodies acquire with the rotation of the axes of the system.

Finally, the point S of the Earth rotates with $\frac{d\vec{R}}{dt} = \vec{\omega}x\vec{R}$ and even more $\frac{d^2\vec{R}}{dt^2} = \vec{\omega}x\left(\vec{\omega}x\vec{R}\right)$. Bringing all pieces together we have:

$$\ddot{\vec{q}} + \left(\dot{\vec{\omega}}x\vec{q}\right) + 2\left(\vec{\omega}x\dot{\vec{q}}\right) + \vec{\omega}x(\vec{\omega}x\vec{q}) = \vec{g} - \vec{\omega}x(\vec{\omega}x\vec{R}) \tag{6}$$

In the [1,2,3] above equation, since $|\vec{q}| \ll R$ and $\left|\dot{\vec{q}}\right| \ll V_e$ the Coriolis term $2\left(\vec{\omega}x\dot{\vec{q}}\right)$ and the centrifugal term are much smaller than $\vec{\omega}x\left(\vec{\omega}x\vec{R}\right)$ and since $\dot{\vec{\omega}} = 0$ (constant angular velocity of earth) we can write:

$$\ddot{\vec{q}} \approx \vec{g} - \vec{\omega}x\left(\vec{\omega}x\vec{R}\right) \tag{7}$$

Now we have $\vec{g} = -g_0\widehat{Z'}$ and

$$\vec{\omega}x\left(\vec{\omega}x\vec{R}\right) = \omega^2 R \;\; \hat{Z}x\left(\hat{Z}x\widehat{Z'}\right) \text{ where } \hat{Z} = C_0\widehat{Z'} + S_0\widehat{y'}$$

$C_0 = \cos\theta_0$, $S_0 = \sin\theta_0$. See Figure III

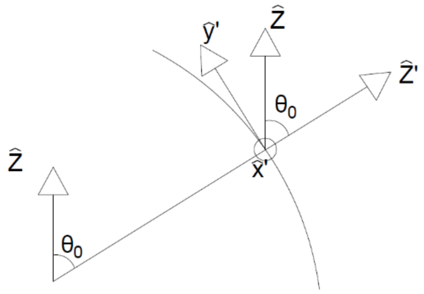


Figure III

In the local non-intentional frame of reference, we have $\hat{Z}x\left(\hat{Z}x\widehat{Z'}\right) = C_0S_0\widehat{y'} - S_0^2\widehat{Z'}$

CENTER OF EARTH

and now eq. (7) becomes

$$\ddot{\vec{q}} = -(g_0 - S_0^2\, g_\omega)\widehat{Z'} - g_\omega S_0 C_0\widehat{y'},\; g_\omega = \omega^2 R \tag{8}$$

ie

$$\ddot{q}_{x'} = 0, \ddot{q}_{y'} = -g_\omega S_0 C_0, \ddot{q}_{z'} = -(g_0 - g_\omega S_0^2) \tag{9}$$

with the initial conditions of a freely falling (heavy) object

$\dot{q}_x'(0) = \dot{q}_y'(0) = \dot{q}_z'(0) = 0$ and

$q_x'(0) = q_y'(0) = 0, q_z'(0) = h$

For simplicity, we drop the primes and the subscripts since we are now sure that we are talking for the local non-inertial frame of reference and by solving (9) we get

$$x=0 \tag{10a}$$

$$y = -{}^{1}\!/_{2}\, C_0 S_0 g_\omega t^2 \tag{10b}$$

$$z_{(t)} = h - {}^{1}\!/_{2}\, (g_0 - g_\omega {S_0}^2) t^2 \tag{10c}$$

If t_f is the total time for the body to reach the ground, then

$$\Delta_y = {}^{1}\!/_{2}\, C_0 S_0 g_\omega {t_f}^2 \text{and } h = {}^{1}\!/_{2}\, (g_0 - g_\omega {S_0}^2) t_f^2$$

which implies

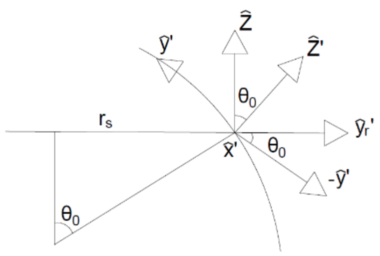
$$\Delta_y = h\, \frac{C_0 S_0 g_\omega}{g_0 - g_\omega {S_0}^2} \tag{11}$$

i.e. a heavy object falling from height h has a deviation <u>to the south</u> given by formula (11) given that h << R and $\sqrt{2gh} << \omega R = V_e$ which is valid for instance for a height h ≈ 10 m. With $g_0 = 9{,}81 \frac{m}{sec^2}$ and $g_\omega = \omega^2 R = 0{,}336930155\ \frac{m}{sec}$ and $\theta_0 \approx 45^o$ we get $\Delta_y = 1{,}7202330956\ cm\ \approx 1{,}720\ cm.$ This deviation is calculated from the vertical position i.e. the line that joins the initial position with the center of the earth (C') and not from the perpendicular plumb line.

For the deviation of the plumb line, see below:

The heavy object that keeps straight the plumb line is motionless in the local non-inertial frame of reference but does a circular motion on radius r_s approximately $r_s \approx R \sin\theta_0$ in the rest frame of the center of earth. (See Figure 4)

The "real" forces exerted on the heavy object are the force $\vec{T}$ that keeps the plumb straight and of course the gravitational force $-mg_0\hat{z}'$. Since the heavy object prescribes a circular motion, we can write:



<u>**Figure IV**</u>

$$m a_S \hat{y}_r = \vec{T} - mg\hat{z}' \tag{12}$$

where $a_s \approx \omega^2 r_s, r_s \approx R\sin\theta_0 = RS_0$

$$\hat{y}_r = -\widehat{y'} C_0 + \widehat{z'}\ S_0 \text{ (See Figure IV)}$$

Substituting $\hat{y}_r$ into (12) we get:

$$\vec{T} \approx -mg_\omega S_0 C_0 \widehat{y'} + (mg_0 + mg_\omega {S_0}^2)\hat{z}' \tag{13}$$

(i.e.) $T_{y\prime} = -mg_{\omega}S_0C_0$ (to the south) and $T_{z\prime} = -m(g + g_{\omega}{S_0}^2)$

Drawing the plumb line (Figure V) we get

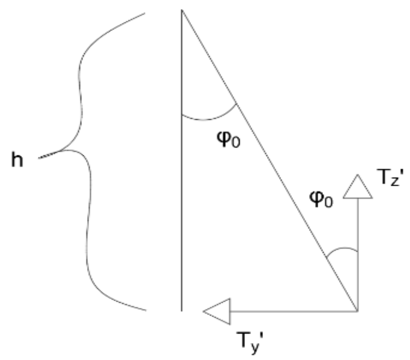


Figure V

$$\tan\varphi_0 = \frac{|T_{y\prime}|}{|T_{z\prime}|} = \frac{g_{\omega}|S_0C_0|}{|g_0+g_{\omega}{S_0}^2|} \tag{14}$$

For $\varphi_0 = 45^{\circ}$, $\tan\varphi_0 = \frac{1/2g_w}{g+1/2g_{\omega}} = 0{,}0017143349$

Since $\Delta_{y'} \approx h\tan\varphi_0$ $(h = 10m = 10^3cm) \rightarrow$

$$\Delta_{y\prime} = 1{,}7143349cm \approx 1{,}714cm$$

The difference between the two deviations (free fall & plumb line) is

$$\Delta^2 y = \Delta_y - \Delta_{y'} = 0{,}6\ mm\ both\ to\ the\ south.$$

CONCLUSIONS

If one leaves a heavy object from the top (A) h=10m of a "perpendicular" plumb line, this will deviate at the bottom about 0,6 mm, which may not even be detectable. The real deviation of the plumb line (due to the rotation of earth) (of h = 10 m) is 1,714 cm from the respective line AC (c Center of Earth) and of the freely falling body 1,720 cm (both to south). A comparison of the two results may not be allowed since the two cases, come from approximate theories, and therefore their difference is very small.

ACKNOWLEDGMENTS

Thanks a lot, to Myrto Saitaki and Dimitris Lamas for printing the manuscript.

REFERENCES

1. T.W.B. Kibble, Classical Mechanics
2. L.D. Landau, Mechanics
3. H. Goldstein, Classical Mechanics