

Geometric Transformations in the Plane: Shapes in Motion

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ABSTRACT

This work intends to analyze the validity of inserting the study of Geometric Transformations in Basic Education. It is based on the analysis, from the historical perspective, of the use of these transformations by man and the development of these concept by mathematicians. It intends to show that this subject can be used as an integrating content between the branches of mathematics in basic education to contextualize teaching of subjects in which students do not easily see a practical application.

Keywords: Geometric Transformations in the Plane; Mathematics; Basic education.

INTRODUCTION

The teaching of geometric transformations in the plane (GTP) is stimulated by Brazilian federal government educational policies since the 1990s. National Curriculum Parameters (PCN) recommends teaching GTP since the first stage of Elementary Education (EE). This document argues that activities involving GTP "allow the development of geometric concepts in a meaningful way" [1, p. 124], develop the child's visualization ability and give a more dynamic perspective to geometry teaching.

In the second stage of the EE "the study of isometric transformations is an excellent starting point for the construction of the notions of congruence" [1, p. 124], as the study of homothety is for the concept of similarity. Also, GTP helps to build a contextualized approach in geometry teaching, by observation of everyday objects such as carpets, ceramics, fabrics, tiles, company logos, window frames, decorative friezes, images reflected in mirrors, etc.

PCN also states that one of the goals of Mathematics for Elementary Education 2nd cycle (EE fourth and fifth years) is to "identify characteristics of geometric figures, discovering similarities and differences between them, through composition and decomposition, symmetries, enlargements and reductions" [2, p.81].

Van de Walle [3, p. 462] defends that in this level of study simpler terms should be used to introduce the basic concepts of GTP. For example, call translation transformation “sliding”, reflection transformation “mirroring” or “turning” and rotation transformation “spin”.

PCN for 6th to 9th grades of Elementary School, associates GTP to concepts and procedures of space and form. This document defends working with

“transformations of figures in the plane through reflections, translations and rotations, and identifying measures that remain invariant in these transformations (measurements of sides, angles, surface). Enlargement and reduction of plane figures according to a ratio and identification of the elements that do not change (measurement of angles) and those that change (measurements of sides, perimeter and area)” [1, p. 73].

It is interesting to show, for example, the sign rule of addition and multiplication operations with integers, content of the seventh year of EE, as geometric transformations on the numbered line (translations, homotheties and reflections). This view of associating operations in a numeric set to "representation of geometric actions" is defended by Mathias [4] also for working with real numbers and, "through the extension of such actions, to build the concept of complex numbers."

In high school, GTP can be used in the graphic analysis of function coefficients, in the exemplification of function itself, bijectivity and composition of functions, in matrix operations, in addition to the construction of the concept of complex numbers.

Geometric transformations may also improve connection between the branches of Mathematics (geometry, algebra, and arithmetic) and between this Science and other domains of knowledge. Through the analysis of crystals, ceramics or other artistic expressions, these connections can be established, "being the starting point for interdisciplinary projects where mathematics, in general, and geometry, in particular, assume important roles" [5, p. 9].

Bastos also defends the use of GTP in basic education for its role in history of Mathematics.

“[...] it would be justified to give much greater importance to geometric transformations, firstly by the relevance they have in the history of recent mathematics - see Félix Klein's Erlangen Program, which influenced the development of mathematics in the 20th century - but also because they are a rich field of connections, a particularly useful tool for demonstrations, for solving problems and, in general, for reasoning about plane and space [6, p. 23].

Thus, GTP permeates all levels of basic education, integrating and contextualizing multiple mathematical contents.

A BIT OF HISTORY

Geometric transformations are present in the constitution of most living beings. For this reason, using symmetries, rotations, translations and homotheties (enlargements and reductions),

copying what is presented by nature, to decorate ceramics, fabrics and baskets is an ancient human practice.

Even though it is impossible for archeology to date the exact moment of the appearance of fabrics and ceramics, it can be validly assumed that they are as old as humanity, after all,

"[...] man was not born equipped with organs to ensure his food, to avoid danger, to maintain body temperature, nor with any special instinct to remedy these deficiencies. The biological success of man in the struggle for survival was achieved by his ability to make instruments, clothing, houses, as exemplified by archaeological record. This ability was learned through experience and error, but in almost all current cases it has been derived from society through accumulated social tradition" [7, p. 34].

At the archeological site of *El Buey*, located between Cochabamba and Santa Cruz, in Bolivia, there is the rock painting reproduced below (Figure 1)



Figure 1. Rock painting of the site of *El Buey* in Bolivia [8]

The Bolivian Rock Art Research Society (SIARB) informs

*"The geometric iconography of these paintings is highly developed and possibly had their inspiration in drawings of fabrics and ceramics. Researcher Roy Querejazu Lewis concluded that *El Buey's* paintings correspond to different periods of the area's agricultural culture" [8].*

The patterns shown in the upper left corner of Figure 1 show very elaborate translations and symmetries and the snake drawn closer to the center of the image is an example of reduction.

In Japan, *Jomon* people created some of the oldest ceramic pieces in the world, dating back to 14,000 BC, and in the period between 4000 BC and 2000 BC these ceramics contained complex and very elaborate decorations.

Chinese pottery, which dates to Neolithic period (3000 BC), was used as a container to store food or funerary or religious rituals. According to the Macao Scientific and Cultural Center (CCCM) [9],

there has been a great improvement in the manufacture of ceramics in China since the invention of the miller wheel (2000 BC), which the Chinese claim for themselves. Figure 2 shows the reproduction of a ceramic bowl from this period, where it is possible to verify the presence of the use of geometric transformations in its decoration.



Figure 2. Painted terracotta platter (3000 BC - 1500 BC) [10]

In Egypt, in the same period as in China's, ceramics were also used for making necklaces, statuettes and amulets. In addition to these people, it is worth mentioning Babylonians and Assyrians, who used decorated ceramic tiles (6th century BC) and Persians.

In Brazil, *Marajoara* ceramics, which represent the artistic production of the inhabitants of Marajó Island, in Pará, are considered one of the oldest ceramic arts in the Americas. It is characterized by the wide and sophisticated amount of ritual, utilitarian and decorative objects. Its ornamentation was more usually done through geometric symbols and symmetrical patterns. Figure 3 shows a photograph of a *marajoara* decorative plate where these characteristics can be observed.



Figure 3. Marajoara ceramic plate (owned by authors)

This plate was purchased in Belém, capital city of Pará state, in 1976, which shows that "the symmetrical traces and colors of *Marajoara* decoration can be found today in local handicrafts in Belém and Marajó Island" [11].

In weaving, symmetries appear in fabrics, carpets, and basketry. Its historical origins are difficult to locate, but it is known that it arose in several places, about the same time. People who inhabited Mesopotamia, Egypt, Greece, Rome, Persia, India, China, and Africa developed their own techniques using cotton, wool, linen, silk, or other fibers.

The image of the Pazyryk carpet (Figure 4), found in the valley with this name in Siberia, is shown below. A carbon-14 dating analysis showed it was woven in the 5th century BC and is considered the oldest rug in the world. Its dimensions are 2.83m by 2m and it is possible to observe the geometric patterns and symmetries in its ornamentation despite wear.



Figure 4. Pazyryk Rug - 5th Century BC [12]

Turkey, Iran, and other Middle Eastern countries still maintain the tradition of making rugs with elaborate geometric motifs. These are the well-known Persian rugs.

Gerdes [13] brings from Mozambique the basketry of crisscrossed handbags, called sipatsi (singular: gipatsi) in the language spoken in the province of its origin (Inhambane). In his book, Gerdes [13, p. 9] presents several different decorative motifs and says that "[...] the collection organized so far shows the existence of more than a hundred motifs, revealing the strength of imagination and artistic creativity and geometric composition of baskets and baskets manufacturers of sipatsi".

Gerdes also relates Mozambique's basketry with similar patterns in Mexico - Oaxaca wallets (Figure 6), which maintains the pre-Hispanic tradition of manufacturing, and in Brazil, in the bags made by the Aracati Indians in Ceará (Figure 7).

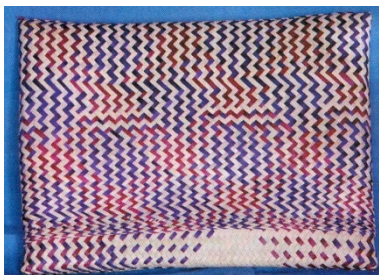


Figure 5: Sipatsi wallet [13 p. 148]



Figure 6: Oaxaca wallet [13, p. 161]



Figure 7: Cear  bag [13, p. 164]

An example which beautifully illustrates the use of geometric transformations in the development of art in weaving comes from the *Arica* culture, located in the northern region of Chile. Bassanezi and Faria [14] show the relationship between the degree of complexity of the ornaments and the organization of the social structures of this culture. They analyze their historical evolution and classify them based on the mathematical study of their drawings.

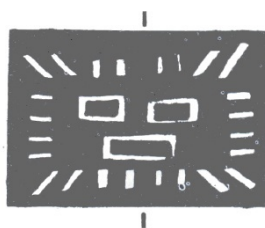


Figure 8: Rosette of the formative period [14, p.43]

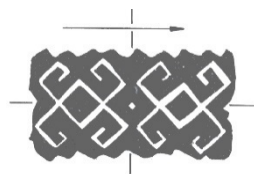


Figure 9: Tiwanaku period bar [14, p.43]



Figure 10: Mosaic of the regional development period [14, p. 44]

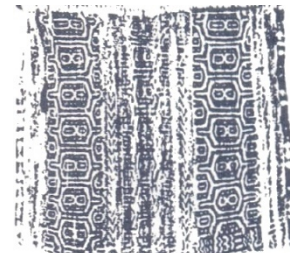


Figure 11: Mosaic from the Inca period [14, p. 44]

It is in the Formative Period (1000 BC - 300 AD) of this culture that the first fabrics adorned with rosettes and barred in caps and blankets appear (Figure 8). These "were produced, in the bars, by translations and reflections perpendicular to the direction of translation and, in the rosettes, by reflection" [14, p. 43].

In the Tiwanaku Period (300 AD – 1100 AD), ornaments gain vivid colors, more elaborate motifs and symmetries produced by rotations, on the rosettes, and translations and reflections perpendicular to the direction of translation, as in the previous period, in addition to reflections over axis in the direction of translations, in the bars. In this period, the first mosaics appear (Figure 9).

In the Regional Development Period (1100– 1470), These people began to use, in addition to colorful and elaborate mosaics, stylized figures of animals and symmetries produced by reflected translations (Figure 10).

In the Inca Period (1470– 1535), despite the advancement of weaving techniques brought by the Inca people, there was no development in terms of ornamentation of the fabrics (Figure 11).

Bassanezi and Faria [14, p. 45] claim that "from an artistic (and mathematical) point of view the Inca period seems to represent an involution".

Coming to modern times, the art done by Dutch artist Maurits Cornelis Escher widely uses geometric transformations. Although his work is supported by mathematical concepts and all of them have a strong mathematical content, he considered himself an absolute layman on the subject.

"It is said that H.M.S. Coxeter, one of modern geometry exponents, enthusiastic about the artist's drawings, invited him to participate in one of his classes. Total failure. To the professor's disappointment, Escher did not know what he was talking about, even when he talked about theories that the artist intuitively applied in his prints" [15].

Escher devoted himself to works with decorative isometries from his visits to the Moorish palace of Allambra, in Granada - Spain. This was built by the Arabs in the 13th century. Even if he did not have the necessary knowledge of geometry, the same cannot be said about these people, famous for their architecture and art.

During these visits, Escher obsessively copied the ornaments on the walls of this palace, and, through these copies, he discovered the movements used in those ornaments: isometries, that is, translation, rotation, reflection, reflected translation and their combinations. Figure 12 shows some of the copies made by Escher in Allambra.



Figure 12: Allambra maiolicas [16, p. 57]

Escher managed, through experimentation, to find out the seventeen existing groups of isometric combinations that leave a certain figure invariant. According to Sérgio Alves, professor of geometry at the Institute of Mathematics and Statistics (IME), University of São Paulo (USP),

"It is remarkable that Escher, without any prior knowledge of mathematics, discovered all these possibilities. As for the four movements, they are the only ones that can be applied on a flat surface so that the result obtained is exactly the original figure. In mathematical terms, they are the only isometries in the plane. The study of these movements is called the Geometry of Transformations and their laws govern the construction of periodic drawings" [15].

Escher called these works the Regular Plane Division (RPD). Some are shown below (Figures 13 to 16).

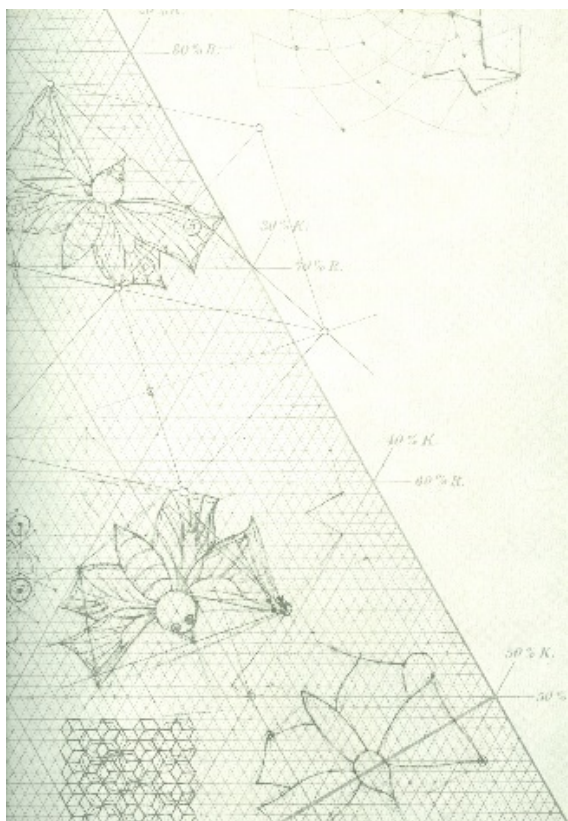


Figure 13: RPD 70 study [16, p. 60]



Figure 14: RPD 70 detail [16, p. 61]

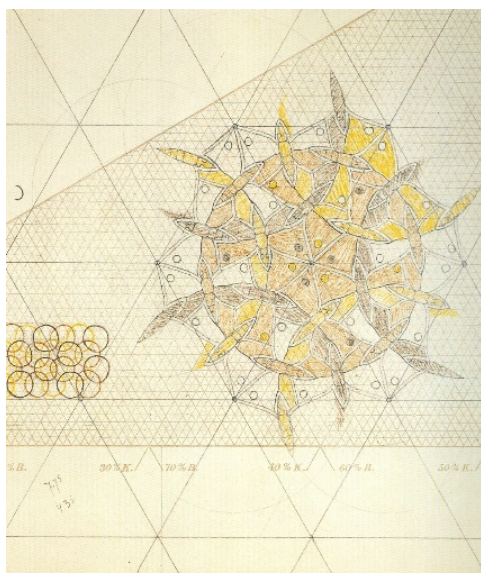


Figure 15: RPD 79 Study [16, p. 76]



Figure 16: RPD 79 detail [16, p. 77]

In nature, in addition to appearing in the constitution of several living beings, geometric transformations can be observed in the composition of the crystals. A crystal is made up of

millions of equal molecules that, when ordered side by side, generate almost perfect symmetrical shapes

"It is possible to assign to each crystalline compound a group of symmetries, to be able to differentiate them well. There are thousands of them in nature. The way to do this consists of starting from a basic figure, the cell, formed by a certain combination of molecules and copying this cell into space, as a reflected, rotated or translated image of the original" [17, p. 150].

Recently, much has been said about quasicrystals and their discoverer, Israeli physicist Daniel Shechtman, who won 2011 Nobel Prize in Chemistry. Until his discovery, it was believed that the atoms of any solid matter were grouped in crystals with patterns of rotational symmetry of four or six axis. Other patterns, such as five, seven, ten or twelve axis, were considered impossible.

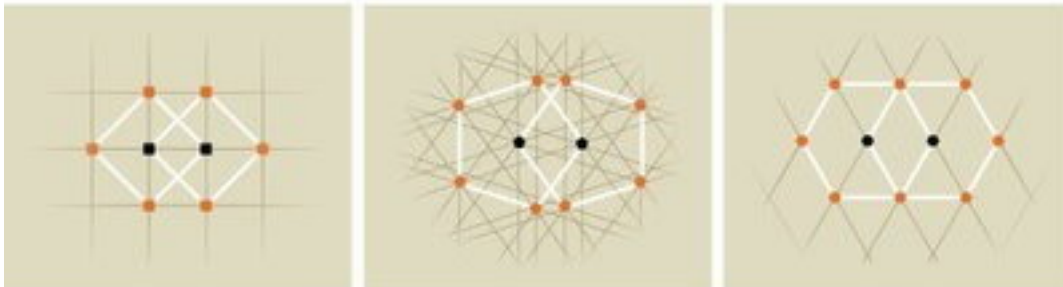


Figure 17: Patterns with four (left), five (center) and six (right) symmetry axis [18]

Figure 17 shows the patterns with four and six axes of symmetry that generate periodic repetitions of the crystal structure, in addition to the five (center) which is a quasicrystal.

In April 1982, Shechtman, through an electronic microscope, visualized atoms arranged in patterns with ten-axis symmetry. "The solid material, synthetic and composed of aluminum and magnesium, resembled, at the molecular level, Arab mosaics, which obey mathematical rules, but do not repeat patterns at any time." [18].

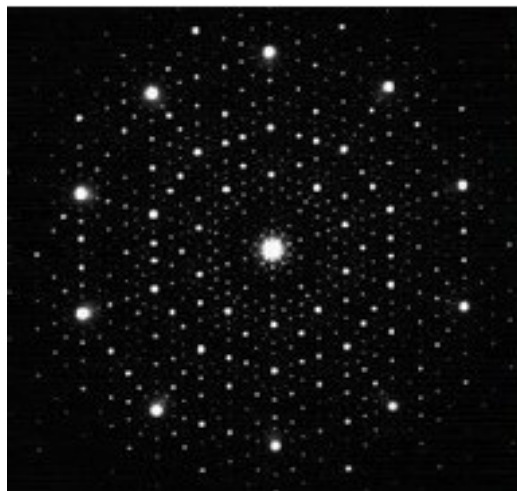


Figure 18: Image observed by Shechtman [18].

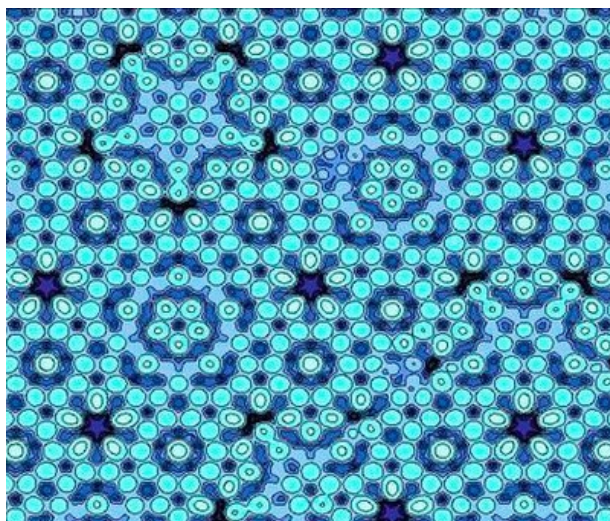


Figure 19: Atomic model of a silver and aluminum quasicrystal [18]

The images shown in Figures 18 and 19 appeared in *O Globo* newspaper, in a journalistic reportage entitled, very suggestively, "Science that looks like art takes Nobel Prize in Chemistry".

"A discovery that brought science and art even closer together and revealed the existence of a structure considered impossible gave the scientist Daniel Shechtman, 70, of the Israel-Technion Institute of Technology, the 2011 Nobel Prize in Chemistry, announced yesterday" [19].

Quasicrystals are present in some of the strongest metal alloys currently used in the manufacture of blades, surgical material, pans lining, as a substitute for Teflon, materials resistant to high temperatures, motors, among others.

In 2009, a mineral composed of aluminum, copper and iron was found in Russia, with ten-axis symmetry patterns. This was the first time that scientists reported the existence of quasicrystals in nature.

So far, this paper has shown that, regardless of mathematization, geometric transformations exist in nature and have been used by man since the beginning of civilization.

This would be an example of what Eves [20, p. 2] calls "subconscious geometry", which "was used by primitive man to make decorative ornaments and drawings, and it is probably correct to say that primitive art prepared the path to later geometric development".

The history of geometric transformations as mathematical content is, therefore, extremely linked to the history of the development of geometry as an area of science. According to Eves [20, p. 1] "nobody ignores that geometry must have probably started in very remote times in antiquity, from very modest origins, then it gradually grew until it reached the enormous dimension that it has today."

It is believed that the origin of geometry is linked to human capacity to recognize and compare shapes and sizes and the practical requirements of using this capacity. For example, the need for land delimitation led man to develop the geometric concept of distance, which led to figuring out simple geometric figures such as rectangles and the triangles. Concepts about position (parallelism, perpendicularism, etc.) were assimilated from the construction of walls and houses. Later, mankind became able to perceive certain general properties of shapes and spatial relationships when comparing objects. Again Eves [20, p. 3] clarifies that

"[...] This brought the advantage of ordering practical geometric problems in groups such that problems of a particular group could be solved by the same general procedure. Thus, the notion of law or geometric rule was arrived at. [...] This higher level of geometric development can be called scientific geometry".

Herodotus, in the 5th century BC, defended the thesis that geometry as a science was born in Egypt. Eves [20, p. 3] transcribes the words of this famous historian:

"They said that this king [Sesostris] divided the land among Egyptians to give each one of them a square lot of equal size and imposing an annual tribute payment on them. But anyone stripped by the river from part of his land would have to go to Sesostris and notify him of what had happened. He then would send his men to observe and measure how much the land had become smaller, so that the owner could pay on what remained, in proportion to the total tax. In this way, it seems to me that geometry originated, being later taken to Hellas (Greece)".

The advances in the study of geometry, generated by the Egyptians, Babylonians, Hindus and Chinese, then arrived in Greece (Hellas), where demonstrative geometry would be developed. Euclid's *Elements* introduced a new research method: the axiomatic model.

"The mathematician has two concerns in mind: the concern with the definitions of his mathematical objects and the concern with demonstrating the properties relating to those objects. So, there are two operations essential in the making of any mathematical theory, the operation of defining the concepts and the operation of demonstrating the properties of those concepts. Defining is, delimiting or explaining a concept in terms of others previously defined. Demonstrating a proposition means arguing for the validity of that proposition using valid logic deduction rules, from propositions already demonstrated" [21].

For this, the mathematician accepts some concepts without definition (primitive concepts) and, from these, defines all other concepts, the so-called derived concepts. It is the same with propositions: he accepts some as true and without proof (the common axioms and notions) and demonstrates, from these, all other propositions, the theorems.

The next major transformation in geometry took place when René Descartes and Pierre de Fermat introduced analytic geometry in the 17th century. With analytical geometry, the study of figures and curves was reduced to the analysis of their corresponding equations. Geometry was then

algebraized. About fifty years after Descartes, Newton's *Principia* would further enhance interest in analytical geometry, reducing even more studies about Euclidean geometry.

At the end of the 18th century, Euler explained transformations in the plane analytically and demonstrated that a displacement in the plane was a rotation, or a translation, or even a translation followed by a symmetry. Afterwards, Sophus Lie (1842 - 1899) and Félix Klein (1849 - 1925) who, based on algebra's group theory, created the notion of groups of transformations and corresponding invariants, thus solving the problem of identifying the structure of the set of transformations.

According to Fainguelernt [22, p. 72], "Klein, in 1870, revolutionized the study of geometry, stating that we must understand its context as the study of the invariant properties of figures in the face of a group's transformations".

Klein exposed his ideas in the Erlangen Program in 1872, "where his deep interest in pedagogical issues and his serious involvement in mathematical research were clearly discernible" [23, p. 605]. He stated that every geometry has a particular transformation group, called *main group*. Thus, the "geometric properties are not altered by the transformations of the main group. We can also state it the other way around: geometric properties are characterized by their invariance with respect to the transformations of the main group" [24, p.7]. With this, Klein associated different groups of transformations to their respective geometries.

Perhaps the main innovation brought by Klein was the union of traditional Euclidian geometry with algebra, presenting every problem under both points of view.

Next, this work will focus on the study of plane Euclidean geometry with its groups of transformations (similarities and isometries), represented by homothety, rotation, translation, symmetries (axial and central), as well as transformations composed by more than one of the mentioned.

TRANSFORMATIONS GEOMETRY

Transformation Geometry is a pedagogical theory used to teach geometry, based on the Erlangen Program. Geometric transformations in the plane (GTP) are part of this theory.

To show the importance of the study of GTP in Basic Education, it is necessary to know its geometric, matrix and algebraic representations. Therefore, based on Fainguelernt and Veloso, translation, central symmetry, axial symmetry, rotation, homothety, and their compositions will be discussed.

A geometric transformation in the plane (or $\mathbb{R}^2$) is a two-way function (or application) of $\mathbb{R}^2$ into $\mathbb{R}^2$, where any given point $P \in \mathbb{R}^2$ can suffer a transformation $T(P)$ that returns a point $Q \in \mathbb{R}^2$, which satisfies the following conditions:

if $P \neq Q$, then $T(P) \neq T(Q)$; that is, T is biunivocal (it is also said to be injective).

for a point $Q \in \mathbb{R}^2$, there is a point $P \in \mathbb{R}^2$ which satisfies $T(P) = Q$; in other words, T is an overjective relation.

In short, a transformation T in the plane π is a bijective function $T: \pi \rightarrow \pi$, which associates to each point P in the plane a point on the same plane $Q = T(P)$. The Q point is called the P image by the T transformation.

The geometric transformations in which a figure only changes places, but retains all its intrinsic geometric characteristics, maintains distances, angles, and areas, are called *isometries*.

Arthur Cayley (1821-1895) developed the algebra of matrices in 1857 and associated it with linear transformations of the type: $x' = ax + by$ and $y' = cx + dy$, where a, b, c, d are real numbers, and which can be imagined as applications that take the point (x, y) to the point (x', y') .

Some geometric transformations are also linear transformations and therefore can be symbolized by the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Translation

Translation is a function or transformation of points on the plane, $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$, defined by $T(x, y) = (x + a, y + b)$, where a and b are constants [22, p.75]. It is an isometry, as it conserves distances, angles, and areas. If $a \neq 0$ and $b \neq 0$ (not a null translation) then the translation is not a linear transformation because $T(0, 0) = (a, b) \neq (0, 0)$.

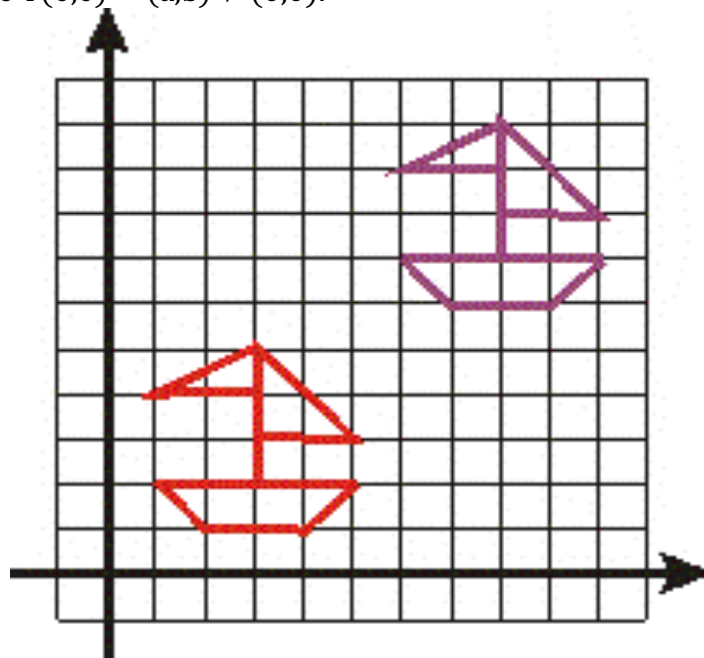


Figure 20: Plane translation representation [22, p. 75]

Symmetry

Symmetry is, geometrically, the exact similarity of the shape in relation to a given line, point or plane. If the figure, or part of it, is reflected, rotated, or moved over the transformed figure, overlapping it, point by point, then it is symmetrical. Symmetry in the plane can be central or axial.

Central Symmetry

The central symmetry in the plane is a function or transformation of points in the plane that associates a given ordered pair (x, y) with its symmetric $(-x, -y)$ [22, p. 74]. The central symmetry applied to a point P of the plane will generate a point P' , such that O (origin) is the midpoint of the segment PP' .

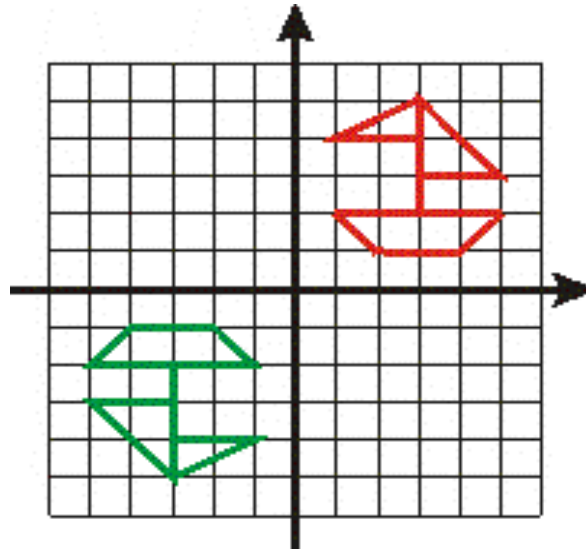


Figure 21: Central symmetry representation [22, p. 74]

Matrix representation of central symmetry

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ -y \end{bmatrix} \text{ the transformation matrix is } \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Observations:

central symmetry is an isometry, that is, it maintains distances, angles, and areas, and therefore the determinant of its transformation matrix is one.

To create symmetry in relation to points outside the origin, simply combine the symmetry with translations: transfer the center of the symmetry to the origin, perform the proposed symmetry and transfer the center of the symmetry back to its initial position.

Axial Symmetry

Axial symmetry in relation to the y axis in the plane is a function or transformation of points in the plane that associates each pair (x, y) with the pair $(-x, y)$ [22, p.74].

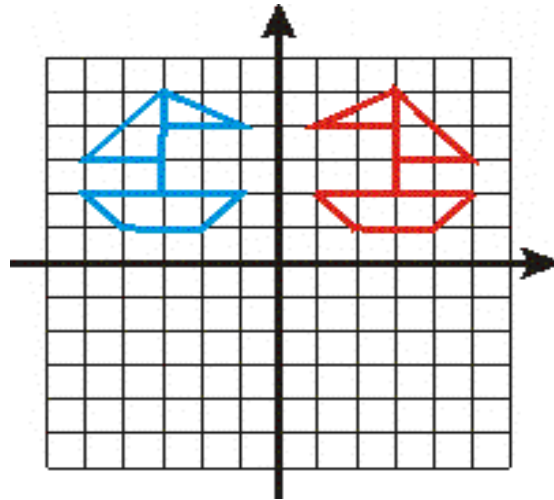


Figure 22: Axial symmetry (y axis) representation [22, p. 74]

Matrix representation of axial symmetry (y axis)

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ y \end{bmatrix} \text{ the transformation matrix is } \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Observations:

Axial symmetry is an isometry, that is, it maintains distances, angles, and areas.

The previous example considers an axial symmetry in relation to the y axis. It can be considered in relation to the x axis, where, if $P = (x, y)$ then $P' = (x, -y)$. In this case the transformation matrix will be $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

Symmetry in relation to any straight line is given by the composition of a symmetry in relation to x axis and a rotation.

Rotation

It is the geometric transformation that performs a rotation, or rotation, of one or more points belonging to a plane π . To be defined, it is necessary to indicate an angle α , which will define the size of the rotation, and a central point, called C, around which the movement will take place.

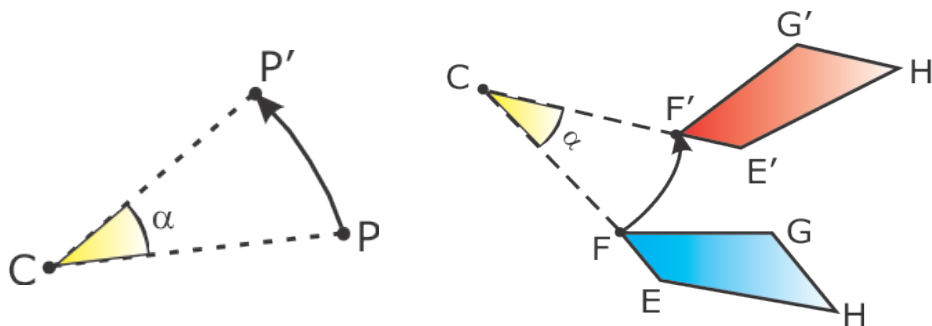


Figure 23: Representations of rotations of a point and a quadrilateral, centered in point C (made by authors)

Matrix representation of rotation

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \cos \alpha - y \sin \alpha \\ x \sin \alpha + y \cos \alpha \end{bmatrix} \text{ the transformation matrix is } \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix}.$$

Observations:

the determinant of the rotation transformation matrix is one, which means that it is an isometry, that is, it maintains distances, angles, and areas. $\begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix} = \cos \alpha \cdot \cos \alpha - (\sin \alpha \cdot -\sin \alpha) = \cos^2 \alpha + \sin^2 \alpha = 1.$

The previous example considers a rotation centered on the origin of the coordinate system. To perform rotations outside this origin, simply combine the rotation with translations: move the center of the rotation to the origin, perform the proposed rotation and move the center of rotation back to its initial position.

symmetry in relation to the origin is exactly equal to a 180° rotation, with center in O.

Homothety

This is a transformation $T: \pi \rightarrow \pi$, centered in the origin O and with a ratio $k > 0$, which associates every point P in plane π to another point $P' = T(P)$, making $OP' = k \cdot OP$.

Considering the definition above, and considering $P = (x, y)$, the coordinates (x', y') of the point P' are given by the system:

$$\begin{cases} x' = k \cdot x \\ y' = k \cdot y \end{cases}$$

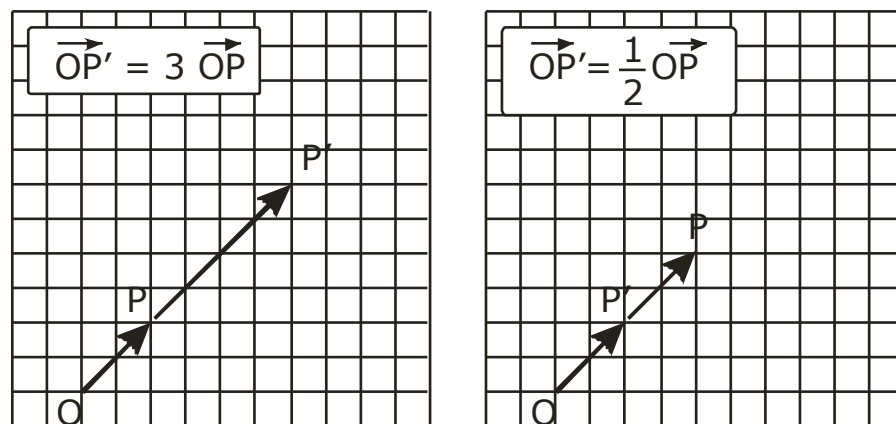
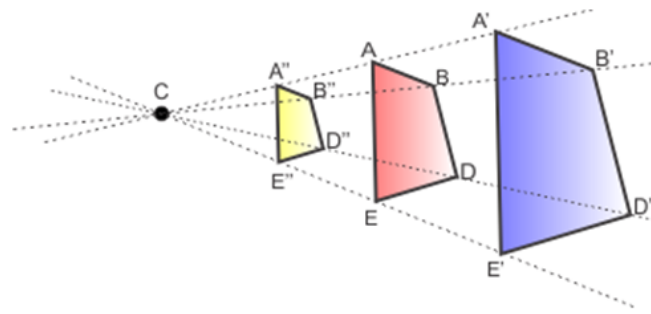


Figure 24: Representation of homothety of an OP-oriented segment (made by authors)

A homothety produces enlargements and reductions in the size of figures in the plane (Figure 24). That is, it generates similar figures based on ratio k.

When considering two similar polygonal figures ABEDF and A'B'D'E'F' (Figure 25), it appears that the corresponding sides of the polygons are proportional to a ratio and that their corresponding angles are congruent.



$$k = \frac{A'B'}{AB} = \frac{B'D'}{BD} = \frac{D'E'}{DE} = \frac{E'A'}{EA}$$

Figure 25: Representation of homothety with center C and ratio k of an ABDE quadrilateral (made by authors)

In the previous example, $A'B'D'E'$ is an enlargement of $ABDE$, with a ratio $k > 1$, and $A''B''D''E''$ is a reduction obtained from a homothety with ratio of $0 < k < 1$. Therefore, to define a homothety, two elements are needed: the center C and the ratio k.

Matrix representation of homothety

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} k.x \\ k.y \end{bmatrix} \text{ the transformation matrix is } \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}.$$

The determinant of a homothety transformation matrix is not one, which means that this transformation is not an isometry, but it maintains the angles of the figures and the proportionality of their sides by k ratio, in addition to the proportionality of their areas in the ratio of the determinant of the transformation matrix.

Observations:

In a similar way as with the rotation with center outside the origin, homotheties with center outside the origin can be obtained through the composition of a translation and a homothety.

Although the definition considers $k > 0$, it is also valid for any $k < 0$. This because a homothety with negative ratio can be obtained through a composition of geometric transformations, that is, a 180° rotation with a homothety of positive ratio (figure 25).

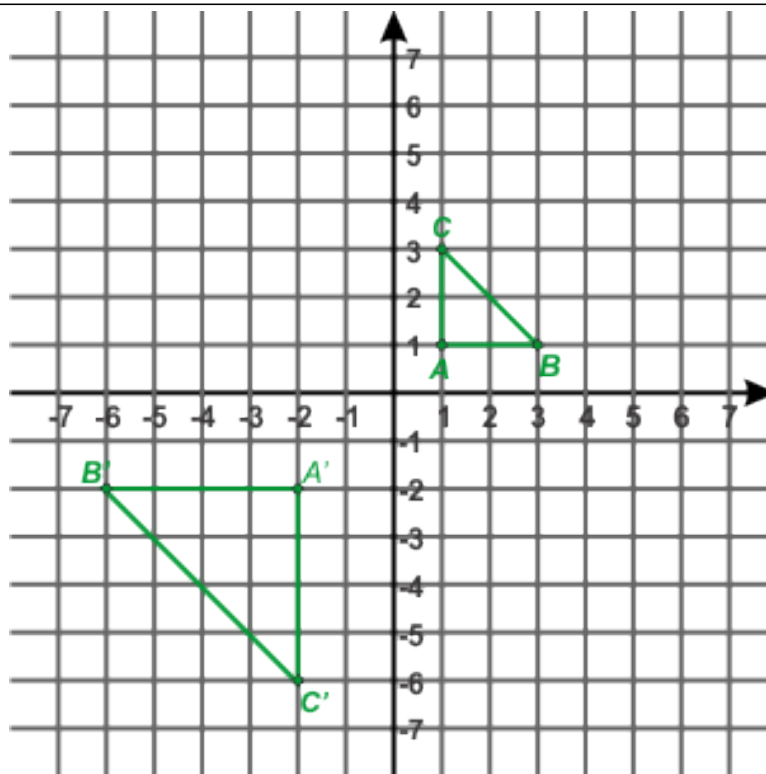


Figure 25: Homothety of $\triangle ABC$ with ratio $k = -2$ (made by authors)

Two homothetic figures are always similar, but two similar figures are not always homothetic. That is, similarity is a necessary condition, but not sufficient to guarantee that a figure is homothetic of another. To check if homothety exists, line segments AA' , BB' and CC' are drawn. If they cross at point O , homothety exists, with that point being its center or focus (Figure 26).

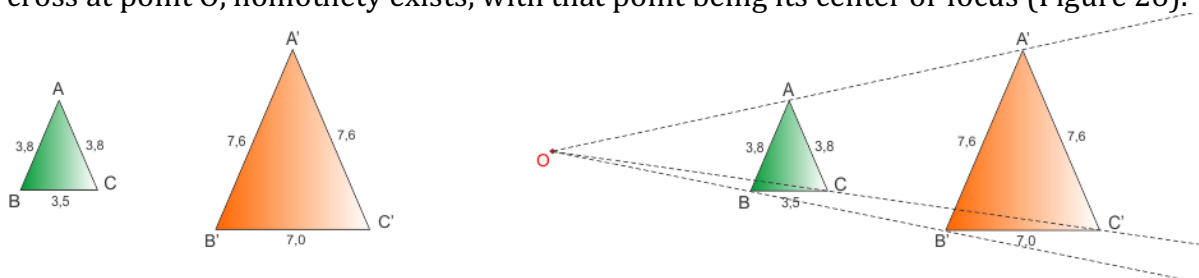


Figure 26: Similar and homothetic triangles (ratio $k=2$, made by authors).

So, for homotetia to exist, it is not enough for the figures involved to be similar, but their corresponding sides must be parallel, and their corresponding vertices must belong to the same line segment. In Figure 27, it is possible to verify that there is no central point, and therefore, there is no homothety.

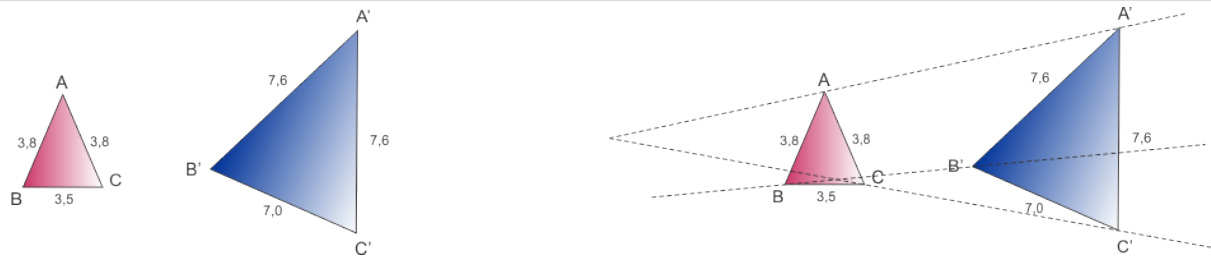


Figure 27: Similar but not homothetic triangles (made by authors).

CONCLUSION

Geometric transformations in the plane introduce an innovative way to teach mathematics in Elementary Education and in High School. By uniting Euclidean geometry with algebra, it shows that the areas of this science are not separated, rather they complement each other.

GTP are also useful to help students to understand concepts like reading maps and working with image enlargements and reductions. They are a powerful tool to show how important mathematics is for arts and architecture, and to show students the importance of geometry throughout mankind's history.

Perhaps their most important contribution in school is to offer new ways to teach mathematical concepts, such as:

- understand congruence and similarity of geometric figures.
- introduce the sign rule for addition and multiplication of whole numbers, presenting them as "representation of geometric actions", enabling a true understanding of these rules and, later, expanding that understanding to the set of real numbers.
- In sequence, explain complex numbers geometric representation in the plane, which will work as a starting point for learning complex numbers algebra.
- contextualize the use of matrix operations and complex numbers algebra and geometric representation in the plane, demonstrating they are different views of the same type of number.

Of the benefits mentioned above, contextualization of operations with matrices and complex numbers to give movement to shapes in video games and animations, is an asset for the basic education math teacher when he needs to answer the questions most raised by his students: why am I studying this subject? Where will I apply it in my life? Surely GTP will provide practical examples for many abstract mathematical concepts.

References

1. BRASIL. Parâmetros Curriculares Nacionais: Matemática - 5ª a 8ª séries. Secretaria de Educação Fundamental. MEC/SEF, Brasília, DF, Brazil, 1998.
2. BRASIL. Parâmetros Curriculares Nacionais: Matemática / Secretaria de Educação Fundamental. 2 ed. DP&A, Rio de Janeiro, RJ, Brazil, 2000.
3. WALLE, J. A. V. Matemática no Ensino Fundamental: Formação de Professores e Aplicação em Sala de Aula. 6 ed. Artmed, Porto Alegre, RS, Brazil, 2009.
4. MATHIAS, C. E. M. Novas Tecnologias no Ensino da Matemática: repensando práticas. Capes/UAB MEC, 2 ed, Brasília, DF, Brazil, 2012

5. BASTOS, R. Simetrias. Notas sobre o Ensino de Geometria. Grupo de Trabalho de Geometria da APM. In Revista Educação e Matemática. APM. Nº 94, May-Jun2006, p. 9-11. Available at <http://www.apm.pt/files/_pp09-11_lq_44f4563b788a8.pdf>. Accessed: 18 dec. 2020.
6. BASTOS, R. Transformações Geométricas. Notas sobre o ensino da Geometria. Grupo de Trabalho de Geometria da APM. In Revista Educação e Matemática. APM. Nº 88, Seep-Oct 2007. p. 23-27. Available at <http://www.apm.pt/files/_23-27_lq_473c3886b161d.pdf>. Accessed: 23 dec. 2020.
7. CHILDE, V. G. Para uma recuperação do passado. A interpretação dos dados arqueológicos. (Tradução de M. Luísa Penafel; estudo introdutório de V. dos Santos Gonçalves). DIFEL, São Paulo, SP, Brazil, 1969.
8. Sociedade de Investigação da Arte Rupestre da Bolívia. Available at <<http://www.siarb-bolivia.org/esp/principal.htm>>. Accessed: 20 aug. 2011.
9. Centro Científico e Cultural de Macau, IP. Available at <<http://www.cccm.pt/page.php?conteudo=&tarefa=ver&id=15&item=Primeiras%20cer%E2micas>>. Accessed: 22 aug. 2011.
10. WIKIPEDIA. Cerâmica Jomon. Available at <http://pt.wikipedia.org/wiki/Per%C3%ADodo_Jomon#cite_note-habu-1>. Accessed: 20 aug. 2011.
11. Itaú Cultural. Arte Marajoara / Cerâmica Marajoara. 2011. Available at <https://enciclopedia.itaucultural.org.br/termo5353/arte-marajoara>. Accessed: 2 jan. 2021.
12. WIKIPEDIA. Tapete Persa. Available at <http://pt.wikipedia.org/wiki/Tapete_persa>. Accessed: 21 dec. 2020.
13. GERDES, P. SIPATSI: cestaria e geometria na cultura Tonga de Inhambane. Moçambique Editora, Mozambique, 2003.
14. BASSANEZI, R.; FARIA, M. S. B. A Matemática dos Ornamentos e a Cultura Arica. In Revista do Ensino de Ciências. Nº 21, 1988. Available at <<https://document.onl/documents/a-matematica-dos-ornamentos-e-a-cultura-da-por-duas-retas-paralelas-de.html>>. Accessed: 2 jan. 2021.
15. LOPES, C. F. Escher o gênio da arte Matemática. 2011. Available at <<http://galileu.globo.com/edic/88/conhecimento2.htm>>. Accessed: 20 dec. 2020.
16. LANDSHOFF, A. (Org). La Magia de M. C. Escher. Edición española. TASCHEN GmbH, Cologne, Germany, 2003.
17. MENDOZA, F. R. Grupos Cristalográficos Planos. In Boletín de la Asociación Matemática Venezolana Vol. VI, No. 1, 1999. Available at <https://www.emis.de/journals/BAMV/conten/vol06/frivero.pdf>. Accessed: 2 jan. 2021.
18. YANO, C. Quasicristais reconhecidos. Ciência Hoje On-line/ PR , 2011. Available at <https://cienciahoje.org.br/acervo/quasicristais-reconhecidos/>. Accessed: 2 jan. 2021.
19. BAIMA, C. Ciência que parece arte leva Nobel de Química. O Globo, Ano LXXXVII, n.28.549, 6 out. 2011, p.36, Rio de Janeiro, RJ, Brazil.
20. EVES H. Tópicos de história da Matemática para uso em sala de aula: Geometria. Atual, São Paulo, SP, Brazil, 1992.
21. BICUDO, I. A Geometria Grega. Palestra proferida em setembro de 2010. Available at <<http://www.youtube.com/watch?v=O3ap76TFG9k>>. Accessed: 20 dec. 2020.
22. FAINGUELERNT, E. K. Educação Matemática: Representação e construção em geometria. Artes Médicas Sul, Porto Alegre, RS, Brazil, 1999.
23. EVES H. Introdução à história da Matemática. Editora da UNICAMP, Campinas, SP, Brazil, 2004.
24. KLEIN, F. O Programa Erlangen. Tradução e comentários Normando Celso Fernandes. Instituto de Física/USP, São Paulo, SP, Brazil, 1984.