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Risky Portfolio Efficient Frontier & Performance: Bayesian versus Frequentist

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ABSTRACT

The location of the Efficient Frontier in Expected Portfolio Return (Rp) / Portfolio Standard Deviation (Sp) space depends on what kind of summary estimates are utilized by the investor. By relying on Bayesian and Frequentist summary measures, in conjunction with a 5-stock portfolio of non-volatility free stocks, analysis in this paper proves that use of reliable Bayesian estimates (estimates that are not sensitive to prior distributions) (a) causes the locus of the Efficient Frontier to appear higher than, and to the right of, the one constructed on Frequentist geometric measures, and (b) generates improved measures of portfolio performance. Additionally, it is demonstrated that the decision of the investor to rely on simple (arithmetic or sample) summary measures describing non-volatility-free assets, would cause the Efficient Frontier to appear above the Bayesian, implying overoptimism. It is concluded that Bayesian estimates, that are not sensitive to mistaken prior distributions, take the investor closer to truer performance levels.

INTRODUCTION

As per collective practical experience, the Mean-Variance Optimization theory of Markowitz (1952, 1959) suffers from several weaknesses. According to Haugh (2016), such weaknesses include "estimation errors in the mean return vector and covariance matrix […], the portfolio weights tend to be extremely sensitive to very small changes in the expected returns […], [and] the presence of heavy tails in the return distributions can result in significant errors in covariance estimates as well." In efforts to deal with such weaknesses, many researchers have established that Bayesian analysis is an effective alternative tool that can be used to deal with parameter uncertainty in portfolio management so that we end up with results closer to the truth. As stated by Avramov et al. (2010) in their abstract, "The Bayesian framework neatly accounts for these uncertainties, whereas standard statistical models often ignore them." Bauder et al. (2018), show the advantages of the Bayesian approach over the Frequentist (or sample) both theoretically and empirically and they conclude that "the constructed Bayesian efficient frontier improves the overoptimism which is present in the sample efficient frontier." Researchers are concerned with the common questions faced by Bayesians such as proper prior distributions, importance of historical data horizons, use of models other than simple mean-variance optimization (such as risk-based or characteristic-based pricing models) and difficulties in data collection in association with these models. See among others, Bawa et al. (1979), Aguilar et al. (2000), Barry (1974), Black et al. (1992), Bodnar et al. (2008, 2009, 2010, 2017), Brown (1976), Frost et al. (1986), Jorion (1986), Kan et al. (2007, 2008), Klein et al.

(1976), MacKinlay et al. (2000), Pastor (2000), Rachev et al. (2008), Sekerke (2015), Stambaugh (1997), Tu et al. (2010).

Based on simple mean-variance optimization, my aim in this paper is to derive a Bayesian Efficient Frontier (BEF) and compare it to the Frequentist Efficient Frontier (FEF) for a 5-stock portfolio. Additionally, I will use the Bayesian and Frequentist results to measure portfolio performance relative to each other and to a market. Assuming that readers are familiar with the Markowitz and Capital Asset Pricing Model (CAPM) models as well as the Sharpe, Treynor, Jensen and M^2 measures of portfolio performance, I will try to re-verify the assertion that the BEF is a better tool to rely on for managing a portfolio. Overall, with the assistance of widely available software statistical packages such as Excel and STATA, I hope to contribute to the basic *pragmatic pedagogy* associated with portfolio efficient frontiers needed by students as well as active investors.

I will proceed as follows: in Section 2, I will deal with the traditional FEF, CAPM, and performance measures. In Section 3, I will deal with the BEF, Bayesian CAPM, and the corresponding performance measures; the analysis will include prediction intervals. In Section 4, I will compare the Frequentist results to the Bayesian results and, thereafter, conclude in Section 5.

Data for my 5-stock portfolio along with a risk-free asset and a market are described in Table 1.

Table 1: Data and Concepts

Time Horizon	Daily rate of return: December 12, 2017 to October 29, 2021 n = number of observations = 219
Stocks	Eli Lilly & Co. (LLY) Walmart Inc. (WMT) Caterpillar Inc. (CAT) Deere & Co. (DE) National Fuel Gas (NFG)
Risk-free Asset rate (R_f)	One-month US Treasury bill rate *
Market Rate (R_m)	Value-weight return of all CRSP firms incorporated in the US * (CRSP = Center for Research in Security Prices)
Fama / French Factors	SMB = return spread of small minus large stocks HML = return spread of cheap minus expensive stocks RMW = return spread of the most profitable firms minus the least profitable CMA = return spread of firms that invest conservatively minus aggressively *

Portfolio (p) and Market (m) Performance Measures	R_p = Expected Portfolio Return S_p = Portfolio St. Deviation S_m = Market St. Deviation β_p = Portfolio Beta Sharpe Portfolio Measure = $(R_p - R_f)/S_p$ Sharpe Market Measure = $(R_m - R_f)/S_m$ Treynor Portfolio Measure = $(R_p - R_f)/ \beta_p$ Treynor Market Measure = $(R_m - R_f)/1$ Jensen Measure (Jensen's alpha) = $\alpha_p = R_p - [R_f + \beta_p(R_m - R_f)]$ Modigliani-squared Measure = $M^2 = (S_m/S_p)(R_p) + (1- S_m/S_p)R_f - R_m$ Portfolio performs better than Market if Portfolio Measures exceed Market Measures. High numerical values of Jensen and M^2 imply good portfolio performance (the higher the better.)

*** Data and definitions:**
Fama & French (1993, 2014, 2015)
French (2022)
<http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html>
Musarurwa, Rujeko, (2019), "Fama French Five Factor Asset Pricing Model", in Quantisti.com
<https://blog.quantinsti.com/fama-french-five-factor-asset-pricing-model/>

FREQUENTIST EFFICIENT FRONTIER AND PORTFOLIO PERFORMANCE

Given that the stocks considered for the portfolio in this paper are not volatility-free, it is more appropriate to utilize, as summary measures, geometric means, variances, and standard deviations; BLOCK 1 of Table 2 displays such summary measures, based on 219 daily rates of return.[1] For the time horizon considered, the values of the R_f are all zero.

The same block displays estimated beta (β) values for each stock as well as the estimated portfolio beta (β_p). The empirical model used for estimation is the conventional, ordinary least squares, 5-factor Capital Asset Pricing (CAPM) model, where t = time in days:

$$(R_{it} - R_{ft}) = \alpha_i + \beta_i(R_{mt} - Rf_t) + \gamma_i SMB_t + \delta_i HML_t + \varepsilon_i RMW_t + \zeta_i CMA_t + e_{it} \quad (1)$$

The multiple regression results, for each stock, are reported in Appendix 1. The entire set and subsets of the independent variables were used, but at the end I selected regressions in which all independent variables considered were significant after correcting for serial correlation. The results passed the F test but suffered from low R^2 (especially the LLY stock) or low $\overline{R}^2$ (all other stocks). I could not reject the $H_0 = 0$ for the intercept coefficients, which is in support of the CAPM. In conjunction with stocks LLY, WMT, DE and NFG, I could not reject the $H_0 \leq 1$ for the beta, which is in some support of the CAPM. For stock CAT, I could not reject the $H_0 = 1$ for its beta, which is in full support of the CAPM. More details from the performed regressions are reported in Appendix 1. My beta estimates are not far from those reported online [see Yahoo

[1] For the computation of geometric summary measures, I used Excel:
=EXP(AVERAGE(LN(…)))-1
=EXP(VAR.S(LN(…)))-1
=EXP(STDEV.S(LN(…)))-1

(2022)], but I trust my estimates more because they are based on data generously provided by French (2022), data that has been recognized as more inclusive and thus more informative.
Table 2 displays results of the Equally Weighted Portfolio (BLOCK 2), Maximum Expected Return Portfolio (BLOCK 3), Minimum Risk Portfolio (BLOCK 4), Tangency Portfolio (BLOCK 5), and the Shorted Portfolio (BLOCK 6).

Expected Portfolio Return (R_p) and Portfolio Standard Deviation (S_p) are computed as suggested by the traditional, widely known and used, Mean-Variance Model of Markowitz (1952). [2]

Using Excel, I computed R_p and S_p as follows: I calculated matrix X (stock rate or return minus corresponding mean), a 219x5 matrix, and then I deposited it along with n in the functions box. In turn, using the command *=MMULT(TRANSPOSE(X),X/(n-1))* I calculated the below Variance-Covariance (VarCov) matrix which I also deposited in the functions box.

Frequentist Variance-Covariance matrix

	LLY	WMT	CAT	DE	NFG
LLY	0.000362	3.79E-05	-7.1E-07	1.46E-05	1.98E-05
WMT	3.79E-05	0.000104	3.86E-05	5.12E-05	3.74E-05
CAT	-7.1E-07	3.86E-05	0.000257	0.000225	9.78E-05
DE	1.46E-05	5.12E-05	0.000225	0.000339	9.96E-05
NFG	1.98E-05	3.74E-05	9.78E-05	9.96E-05	0.000189

Thereafter, I calculated R_p and S_p using the following commands:
R_p = *SUMPRODUCT(…weights… , …means with dollar signs…)*
S_p =*SQRT(MMULT(MMULT(…weights… , VarCov),TRANSPOSE(…weights…)))*

Finally, using Excel's "Solver", I derived, for each BLOCK in Table 2, the weights that optimize R_p and S_p which I then used, in conjunction with the portfolio's beta and the summary measures of market and risk-free rates of return, to calculate the six Performance Measures, defined in Table 1[3]. (For the results in BLOCK 6, "Solver" was set to maximize the Sharpe Portfolio Measure with leaving unchecked the box *Make Unconstrained Variables Non-Negative*.)

[2] For a detailed description of the model, the interested reader may browse and / or download a version of it available by the Hong Kong University of Science and Technology (2022).

[3] The Investor can also use the Sortino Ratio [Sortino et al. (1994)] to measure portfolio performance. Sortino modifies the Sharpe ratio by replacing R_f with a target rate of return and standard deviation with downside deviation (negative returns as its risk measure) in the calculation. A good Sortino Ratio is one with a score of 2 or above. Without a doubt, the Sortion measure, with a target rate of return, would be better than the Sharpe measure, but because I do not plan to specify a target rate of return for my portfolio, which portfolio performance measure is better than another is not my concern.

Table 2: Frequentist Portfolio Geometric Summary and Performance Measures

BLOCK 1		LLY	WMT	CAT	DE	NFG	Market	Risk-free
	Mean	0.001794	0.0001236	0.0005846	0.001237	0.0012883	0.0009626	0
	Standard Deviation	0.0190246	0.0101877	0.0160348	0.0184015	0.0137577	7.055E-05	0
	Variance	0.0003619	0.0001038	0.0002571	0.0003386	0.0001893	0.0083996	0
	Beta	0.4150446	0.5885593	1.104218	1.249646	0.6227529		
	Portfolio Beta	0.7960442						
BLOCK 2	**Equally Weighted**	LLY	WMT	CAT	DE	NFG		
	Weights	0.2	0.2	0.2	0.2	0.2	1	
	Expected Portfolio Return	0.0010055	per Day	0.2882198	per Year			
	Portfolio Standard Deviation	0.0099866						
	Sharpe Portfolio Measure	0.1006858						
	Sharpe Market Measure	0.1146025						
	Treynor Porftfolio Measure	0.0012631						
	Treynor Market Measure	0.0009626						
	Jensen Measure	0.0002392						
	M^2 Measure	-0.000117						
BLOCK 3	**Maximum Expected Return**	LLY	WMT	CAT	DE	NFG		
	Weights	1.0000001	0	0	0	0	1.0000001	
	Expected Portfolio Return	0.001794	per Day	0.5709559	per Year			
	Portfolio Standard Deviation	0.0190246						
	Sharpe Portfolio Measure	0.0942995						
	Sharpe Market Measure	0.1146025						
	Treynor Porftfolio Measure	0.0022537						
	Treynor Market Measure	0.0009626						
	Jensen Measure	0.0010277						
	M^2 Measure	-0.000171						
BLOCK 4	**Minimum Risk**	LLY	WMT	CAT	DE	NFG		
	Weights	0.131422	0.5436624	0.1267643	0	0.1981514	1	
	Expected Portfolio Return	0.0006324	per Day	0.1726943	per Year			
	Portfolio Standard Deviation	0.0085563						
	Sharpe Portfolio Measure	0.0739059						
	Sharpe Market Measure	0.1146025						
	Treynor Porftfolio Measure	0.0007944						
	Treynor Market Measure	0.0009626						
	Jensen Measure	-0.000134						
	M^2 Measure	-0.000342						
BLOCK 5	**Tangency**	LLY	WMT	CAT	DE	NFG		
	Weights	0.3872058	0	0	0.1780884	0.4347058	1	
	Expected Portfolio Return	0.001475	per Day	0.4497945	per Year			
	Portfolio Standard Deviation	0.0111742						
	Sharpe Portfolio Measure	0.1319989						
	Sharpe Market Measure	0.1146025						
	Treynor Porftfolio Measure	0.0018529						
	Treynor Market Measure	0.0009626						
	Jensen Measure	0.0007087						
	M^2 Measure	0.0001461						
BLOCK 6	**Shorted**	LLY	WMT	CAT	DE	NFG		
	Weights	0.543099	-0.392681	-0.337239	0.4833585	0.7034624	1	
	Expected Portfolio Return	0.0022328	per Day	0.7542562	per Year			
	Portfolio Standard Deviation	0.0158028						
	Sharpe Portfolio Measure	0.1412927						
	Sharpe Market Measure	0.1146025						
	Treynor Porftfolio Measure	0.0028049						
	Treynor Market Measure	0.0009626						
	Jensen Measure	0.0014665						
	M^2 Measure	0.0002242						

To draw the 5-stock Efficient Frontier (EF), I used the Excel's command *=RANDARRAY(…)* to collect 10,000 values of weights which in turn were used to calculate the corresponding R_p and S_p values. Figure 1 portrays the EF with R_p on the vertical axis and S_p on the horizontal axis.

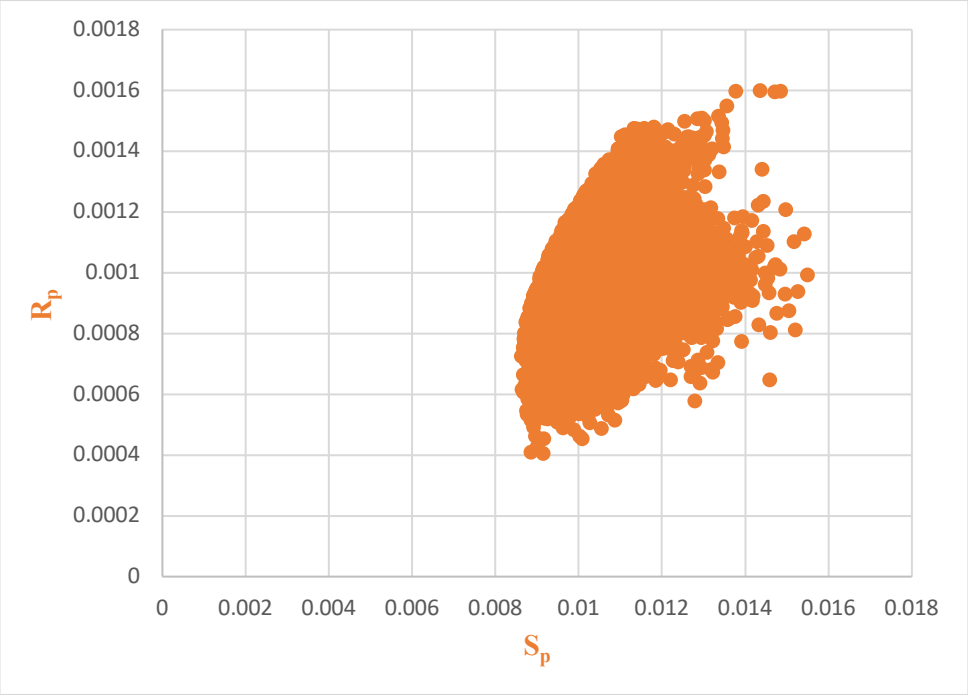


Figure 1: Frequentist Efficient Frontier based on Geometric Summary Measures

Bayesian Efficient Frontier and Portfolio Performance

The Bayesian philosophy may best be expressed by stating the Bayesian Theorem in its most popular depiction: given events A and B, P(A|B) = [P(B|A)P(A)] / P(B), where P(A|B) = posterior probability distribution function, P(B|A) = likelihood probability function, P(A) = prior probability distribution, and P(B) = evidence probability function.

In Bayesian inference, probability is interpreted as a subjective degree of belief and the goal is to state and analyze the belief. Bayesian thinking requires that probability states our degree of belief, a parameter is a random variable, probability statements are about parameters, and frequency guarantees are not offered.

To begin with, the investor ought to start with a subjective prior distribution. "Subjective" can be irrational in which case the analysis will cause error ridden results. Naturally, the investor must aim for rational subjectivity (subjectivity that can be credibly defended) and proceed accordingly. The prior distribution reflects knowledge about a parameter prior to observing data, which, after data is observed, is updated to generate the posterior distribution of the parameter which may be used for Bayesian inference. If the investor does not possess credible prior information, some kind of noninformative prior may be employed. In general, the smaller the size of sample (which by nature contains less information about the parameter under study) the more important becomes the choice of a prior distribution.

More formally, Bayesian inference requires that (a) the investor chooses a prior probability distribution, $\pi(\theta)$, that expresses her beliefs about a parameter θ before she observes any data; (b) chooses a statistical model $p(x|\theta)$ that reflects her probability or belief p about x given θ; and (c) after observing data $D_n = \{X_1, \ldots, X_n\}$, updates her beliefs and calculates the posterior distribution $p(\theta|D_n)$. As per Bayes' theorem, the posterior distribution can be expressed as

$$p(\theta|X_1, \dots, X_n) = \frac{p(X_{1,\dots,}X_n|\theta)\pi(\theta)}{p(X_1, \dots, X_n)} = \frac{L_n(\theta)\pi(\theta)}{c_n} \propto L_n(\theta)\pi(\theta)$$

where $L_n(\theta) = \prod_{i=1}^{n} p(X_i|\theta)$ is the likelihood function and

$c_n = p(X_1, \dots, X_n) = \int p(X_1, \dots, X_n|\theta)\pi(\theta)d\theta = \int L_n(\theta)\pi(\theta)d\theta$

is the normalizing constant, which is also called evidence.

Posterior distributions can be estimated by relying on Markov Chain Monte Carlo (MCMC) sampling. The investor can rely on Bayesian inference, based on the posterior distribution of the parameter under study, to calculate summary measures of this distribution including means and variances and their Monte Carlo Standard Error (MCSE) as well as credible intervals. Additionally, the investor can perform regression analysis (for example, propose and test a Bayesian CAPM.) For detailed descriptions, see Bernardo et al. (2000), Carlin et al. (2000), and Oliphant (2006). In whatever follows, based on Bayesian computational methodology, I calculate means, variances and regressions to produce results similar to those in Section 2 above and then compare the Bayesian-based portfolio to the Frequentist-based one.

Using a STATA algorithm, specifically, the "*bayesmh*" command with flat priors [STATA Manual (2014)], I calculated for each stock the posterior Bayesian estimates of mean, variance and beta from the CAPM - based on empirical model (1) and Bayesian regression methodology. Regarding regression, as in section 2, I run each stock excess return versus all independent variables of the 5-factor model, but at the end I selected to include only those independent variables that contributed to convergence.

Figure 2 displays self-explanatory estimates and diagnostics for LLY. The estimates for mean (.0019514) and variance (.000374) are both greater than the corresponding geometric mean and variance for the same stock reported in Block 1 of Table 2; they are close to those values, as expected, given that I used a flat prior because I did not possess any prior knowledge of the variable under consideration. Based on 10,000 iterations, the low values of MCSE - the lower the better - indicate high accuracy which is good enough. However, a greater sample size could generate higher accuracy. The credible intervals indicate that the probability that the mean of lly is between -.0006971 to .0044932 is about 95% and that the probability that the variance of lly is between .0003112 to .0004467 is about 95% as well. Obviously, knowledge of credible intervals about mean and variance can be helpful in the investor's decision-making process. It must be noted here that because of MCMC, a simulation-based method, the results will be different every time the STATA command "*bayesmh*" is run. I run the command many times and computed the grand averages of the resulting mean and variance values; I observed no significant change regarding convergence. Convergence is confirmed visually: as the graphs indicate, trace plots demonstrate good mixing, the autocorrelation dies off quickly, and the posterior distribution resembles the normal distribution - as is expected for the specified likelihood and prior distributions.

In turn, I estimated the empirical CAPM proposed in (1) for LLY, subject to Bayesian regression methodology. As in section 2, I run the LLY excess return as a function of the entire set and then subsets of the independent variables of the 5-factor model. At the end, out of all regressions, I selected the one of (R_{it} - R_{ft}) versus (R_{mt} - Rf_t) and HML_t because this combination of variables contributed to best convergence. The Bayesian regression results are reported in Figure 3. The MCSEs values are nearly zero and all visual diagnostics confirm convergence. The Bayesian beta

of .0040617 is close to but slightly higher than the corresponding Frequentist beta of 0.0040147; in other words, undiversifiable risk appears as a slightly more serious issue when the investor estimates via the Bayesian approach.

I applied the "*bayesmh*" command subject to normal priors as well, at various reasonable means and variances, but the results were qualitatively similar to those based on flat priors. I proceeded similarly with all other stocks regarding means, variances and betas; all results are available on request.

In turn, applying the same Excel commands as in Section 2, I computed the Bayesian Variance-Covariance matrix, constructed Table 3 for comparison with Table 2, and graphed the Bayesian Efficient Frontier in Figure 4. The results in Table 3 indicate that the Bayesian portfolio preforms better than the Frequentist across the board. Figure 5 graphs together the two frontiers (those in Figures 1 and 4) showing that the Bayesian Efficient Frontier has moved up and to the right: comparing expected portfolio return and portfolio standard deviation values (for example, those associated with the minimum risk portfolio) we see that values under the Bayesian Frontier exceed those under the Frequentist Frontier.

Bayesian Variance-Covariance Matrix

	LLY	WMT	CAT	DE	NFG
LLY	0.000374	3.83132E-05	-7.3456E-07	1.48E-05	1.99E-05
WMT	3.83132E-05	0.0001038	3.88798E-05	5.19E-05	3.78E-05
CAT	-7.34563E-07	3.88798E-05	0.0002592	0.000227	9.83E-05
DE	1.47611E-05	5.18596E-05	0.000226867	0.000347	0.000101
NFG	1.99425E-05	3.78343E-05	9.83227E-05	0.000101	0.000192

```
. bayesmh lly, likelihood(normal({var})) prior({lly:_cons}, flat) prior({var}, jeffreys)

Burn-in ...
Simulation ...

Model summary
------------------------------------------------------------------------------
Likelihood:
  lly ~ normal({lly:_cons},{var})

Priors:
  {lly:_cons} ~ 1 (flat)
        {var} ~ jeffreys
------------------------------------------------------------------------------

Bayesian normal regression                      MCMC iterations  =     12,500
Random-walk Metropolis-Hastings sampling        Burn-in          =      2,500
                                                MCMC sample size =     10,000
                                                Number of obs    =        219
                                                Acceptance rate  =      .2903
                                                Efficiency:  min =      .09742
                                                             avg =      .1126
Log marginal likelihood =   547.8773            max =      .1279
```

	Mean	Std. Dev.	MCSE	Median	Equal-tailed [95% Cred. Interval]	
lly						
_cons	.0019514	.0013055	.000037	.0019255	-.0006971	.0044932
var	.000374	.0000355	1.1e-06	.0003724	.0003112	.0004467

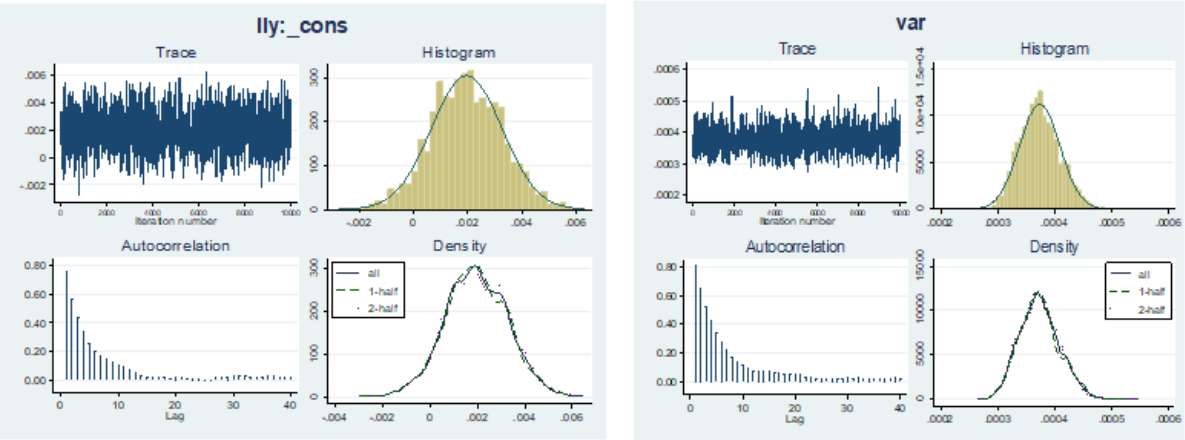


Figure 2: Posterior Bayesian Mean and Variance for LLY

```
. bayesmh lly mktrf hml, likelihood(normal({var})) prior({lly:_cons mktrf hml}, flat) prior({var}, jeffreys)

Burn-in ...
Simulation ...

Model summary
------------------------------------------------------------------------------
Likelihood:
  lly ~ normal(xb_lly,{var})

Priors:
  {lly:_cons mktrf hml} ~ 1 (flat)                                    (1)
                {var} ~ jeffreys
------------------------------------------------------------------------------
(1) Parameters are elements of the linear form xb_lly.

Bayesian normal regression                      MCMC iterations  =     12,500
Random-walk Metropolis-Hastings sampling        Burn-in          =      2,500
                                                MCMC sample size =     10,000
                                                Number of obs    =        219
                                                Acceptance rate  =      .2766
                                                Efficiency:  min =      .03599
                                                             avg =      .0408
Log marginal likelihood =  540.48501                         max =      .04593

------------------------------------------------------------------------------
             |                                                Equal-tailed
             |      Mean   Std. Dev.     MCSE     Median  [95% Cred. Interval]
-------------+----------------------------------------------------------------
lly          |
       mktrf |   .0040617   .0016029   .000075   .0041049   .0009371   .0071274
         hml |  -.0005959   .0012202   .000059  -.0005879  -.0030221   .0017576
       _cons |   .0016313   .0013166   .000067   .0016066  -.0010196   .0041779
-------------+----------------------------------------------------------------
         var |   .0003638   .0000356   1.9e-06   .0003616   .0003014   .0004427
------------------------------------------------------------------------------

. bayesgraph diagnostics {lly:mktrf hml}, histopts(normal)
```

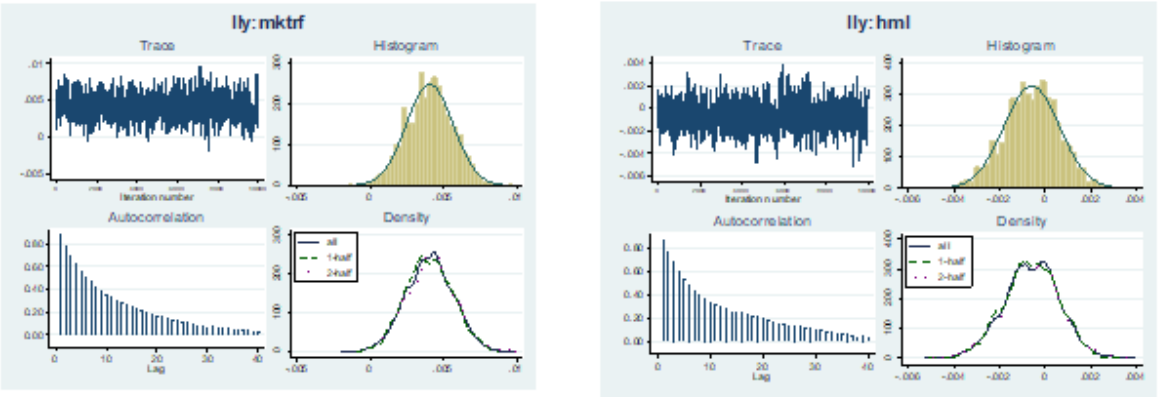


Figure 3: Bayesian CAPM regression results for LLY

Table 3: Bayesian Portfolio Summary and Performance Measures

BLOCK 1		LLY	WMT	CAT	DE	NFG	Market	Risk-free
	Mean	0.0019514	0.0002	0.000684	0.001357	0.001354	0.001016	0
	Standard Deviation	0.01933908	0.010188	0.0161	0.018633	0.01386	0.008444	0
	Variance	0.000374	0.000104	0.000259	0.000347	0.000192	7.13E-05	0
	Beta	0.3774495	0.572781	1.102818	1.21623	0.609866		
	Portfolio Beta	0.77582884						
BLOCK 2	**Equally Weighted**	LLY	WMT	CAT	DE	NFG		
	Weights	0.2	0.2	0.2	0.2	0.2	1	
	Expected Portfolio Return	0.00109988	per Day	0.319189	per Year			
	Portfolio Standard Deviation	0.01005497						
	Sharpe Portfolio Measure	0.10938672						
	Sharpe Market Measure	0.12035861						
	Treynor Porftfolio Measure	0.00141768						
	Treynor Market Measure	0.0010163						
	Jensen Measure	0.00031141						
	M² Measure	-9.265E-05						
BLOCK 3	**Maximum Expected Return**	LLY	WMT	CAT	DE	NFG		
	Weights	0.99999998	0	0	0	0	1	
	Expected Portfolio Return	0.0019514	per Day	0.634396	per Year			
	Portfolio Standard Deviation	0.01933908						
	Sharpe Portfolio Measure	0.10090449						
	Sharpe Market Measure	0.12035861						
	Treynor Porftfolio Measure	0.00251525						
	Treynor Market Measure	0.0010163						
	Jensen Measure	0.00116293						
	M² Measure	-0.0001643						
BLOCK 4	**Minimum Risk**	LLY	WMT	CAT	DE	NFG		
	Weights	0.13051126	0.54533	0.12774	0	0.196419	1	
	Expected Portfolio Return	0.0006914	per Day	0.190262	per Year			
	Portfolio Standard Deviation	0.00856954						
	Sharpe Portfolio Measure	0.08068162						
	Sharpe Market Measure	0.12035861						
	Treynor Porftfolio Measure	0.00089118						
	Treynor Market Measure	0.0010163						
	Jensen Measure	-9.707E-05						
	M² Measure	-0.000335						
BLOCK 5	**Tangency**	LLY	WMT	CAT	DE	NFG		
	Weights	0.39064613	0	0	0.171472	0.437882	1	
	Expected Portfolio Return	0.00158783	per Day	0.491548	per Year			
	Portfolio Standard Deviation	0.01131579						
	Sharpe Portfolio Measure	0.14031951						
	Sharpe Market Measure	0.11573131						
	Treynor Porftfolio Measure	0.00204662						
	Treynor Market Measure	0.0976003						
	Jensen Measure	0.00092709						
	M² Measure	0.0207361						
BLOVK 6	**Shorted**	LLY	WMT	CAT	DE	NFG		
	Weights	0.55873236	-0.42151	-0.3406	0.481999	0.721376	1	
	Expected Portfolio Return	0.00242349	per Day	0.840394	per Year			
	Portfolio Standard Deviation	0.01632678						
	Sharpe Portfolio Measure	0.14843631						
	Sharpe Market Measure	0.12035861						
	Treynor Porftfolio Measure	0.00312374						
	Treynor Market Measure	0.0010163						
	Jensen Measure	0.00163501						
	M² Measure	0.00023709						

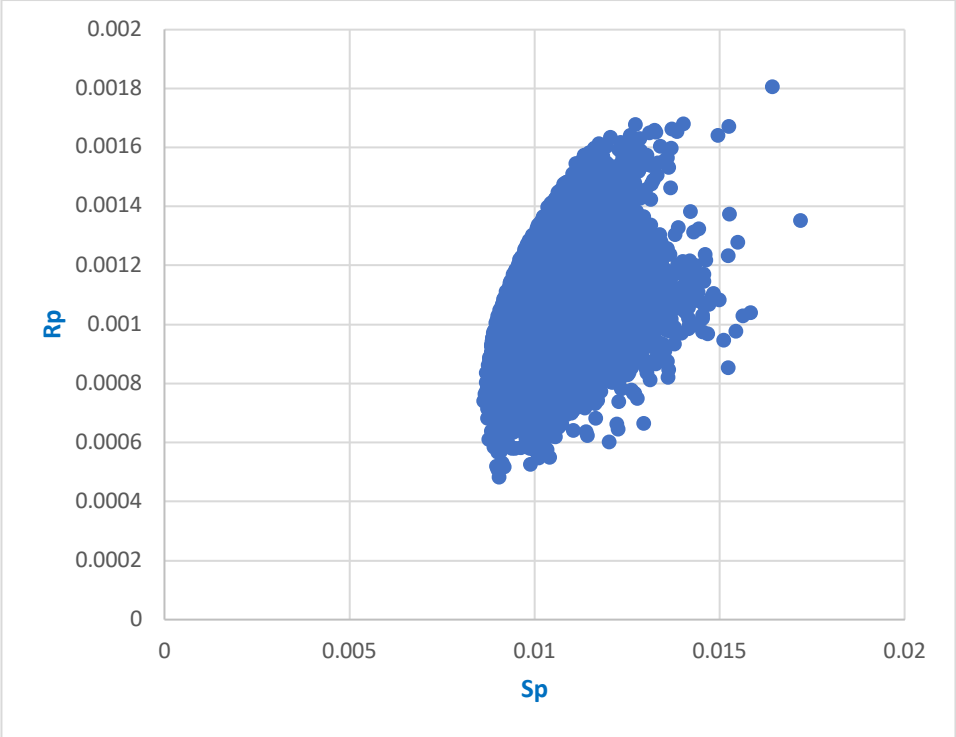


Figure 4: Bayesian Efficient Frontier

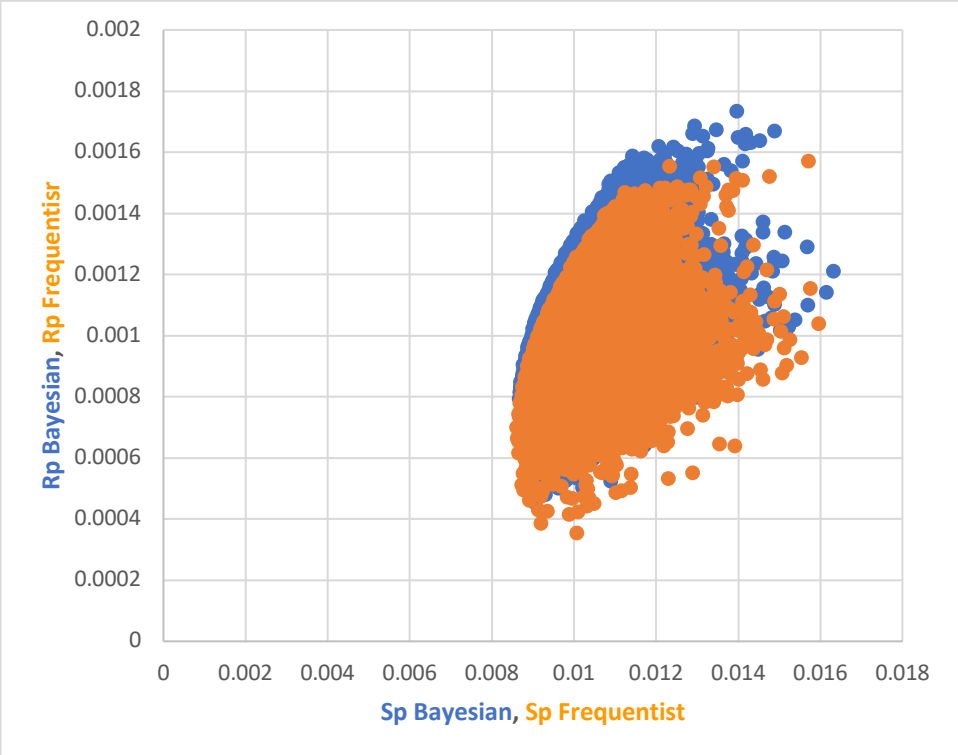


Figure 5: Frequentist and Bayesian (in blue) Efficient Frontiers

Hence, the location of the Efficient Frontier in Rp, Sp space depends on what kind of summary estimates are utilized by the investor. As the above analysis proves, for the time horizon considered, reliance on Bayesian estimates shows a better-performing portfolio (Table 2) than

reliance on frequentist estimates based on geometric methodology (Table 1.) In other words, the Bayesian results take the investor closer to truer results, assuming that results are not sensitive to erroneously chosen prior distributions.

Unfortunately, many investors are taken further away from the truth by relying on arithmetic or simple summary measures which cause the locus of points of the Efficient Frontier to appear above the Bayesian. To demonstrate this point, I used the simple summary measures - those in the 2-row table below - for a 2-stock portfolio (chosen from the 5 stocks used above) consisting of stocks LLY and DE to derive the corresponding Efficient Frontier (**EF_S**); then, for the same portfolio, I derived the Efficient Frontier based on geometric summary measures (**EF_G**), and the one based on Bayesian summary measures (**EF_B**). Figure 6 displays these Efficient Frontiers clearly showing that the use of arithmetic (simple) summary measures contribute to investor overoptimism.

Mean of LLY	Mean of DE	Variance of LLY	Variance of of DE	Covariance
0.0019755	0.001406	0.000370028	0.0003425	0.000015

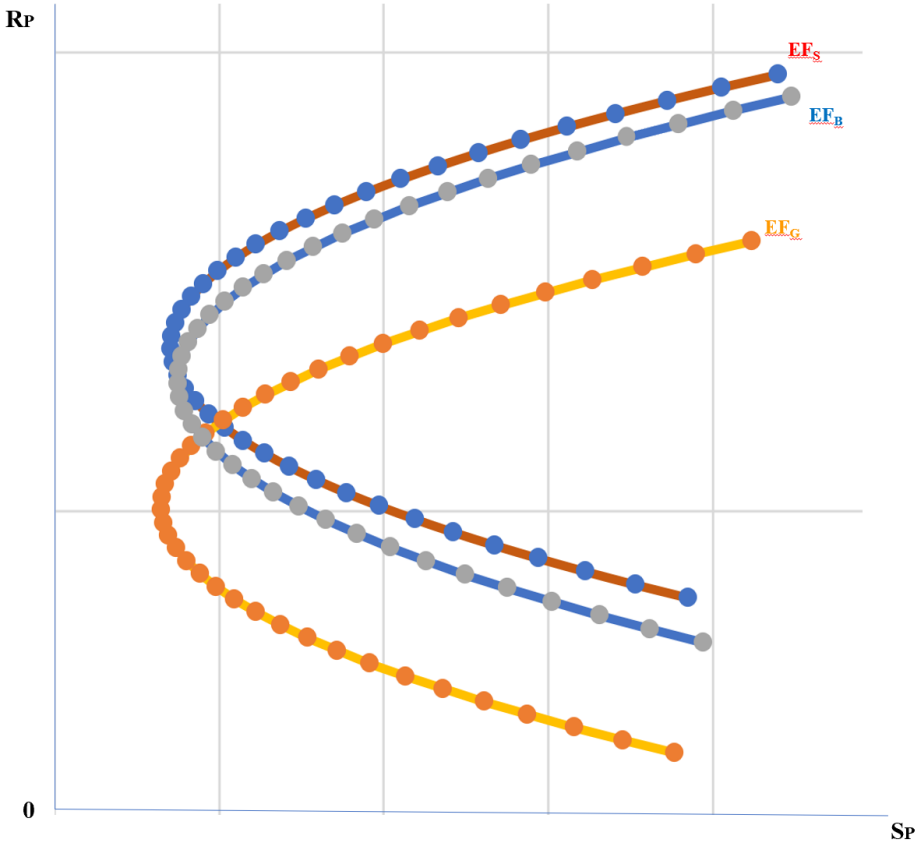


Figure 6: 2-stock Portfolio Efficient Frontiers

CONCLUSION

By relying on Bayesian and Frequentist summary measures, in conjunction with a 5-stock portfolio of non-volatility free stocks, analysis in this paper proves that use of reliable Bayesian estimates (estimates that are not sensitive to prior distributions) (a) causes the locus of the Efficient Frontier to appear higher than, and to the right of, the one constructed based on

Frequentist geometric measures, and (b) generates improved measures of portfolio performance. Additionally, it was demonstrated that the decision of the investor to rely on simple (arithmetic or sample) summary measures describing non-volatility-free assets, would cause the Efficient Frontier to appear above the Bayesian, implying overoptimism. The results are credible because they are not sensitive to prior distributions; as demonstrated, convergence was achieved based on flat or normal prior distributions. The lesson of the analysis is obvious: by relying on credible Bayesian estimates, the investor experiences a better-performing portfolio, closer to truer potential gains.

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APPENDIX 1: CAPM FREQUENTIST REGRESSION RESULTS

```
. prais lly mktrf

Iteration 0:  rho = 0.0000
Iteration 1:  rho = -0.1217
Iteration 2:  rho = -0.1219
Iteration 3:  rho = -0.1219

Prais-Winsten AR(1) regression -- iterated estimates

      Source |       SS           df       MS      Number of obs   =       219
-------------+----------------------------------   F(1, 217)       =      7.49
       Model |  .002649729         1  .002649729   Prob > F        =    0.0067
    Residual |  .076745251       217  .000353665   R-squared       =    0.0334
-------------+----------------------------------   Adj R-squared   =    0.0289
       Total |  .079394979       218  .000364197   Root MSE        =    .01881

------------------------------------------------------------------------------
         lly |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
       mktrf |   .4150446   .1516133     2.74   0.007     .1162214    .7138677
       _cons |   .0015606   .0011433     1.37   0.174    -.0006928    .0038139
-------------+----------------------------------------------------------------
         rho |  -.1218668
------------------------------------------------------------------------------
Durbin-Watson statistic (original)    2.243274
Durbin-Watson statistic (transformed) 1.964547


. prais wmt mktrf rmw

Iteration 0:  rho = 0.0000
Iteration 1:  rho = -0.1057
Iteration 2:  rho = -0.1082
Iteration 3:  rho = -0.1082
Iteration 4:  rho = -0.1082
Iteration 5:  rho = -0.1082

Prais-Winsten AR(1) regression -- iterated estimates

      Source |       SS           df       MS      Number of obs   =       219
-------------+----------------------------------   F(2, 216)       =     29.64
       Model |  .004831967         2  .002415983   Prob > F        =    0.0000
    Residual |  .017608598       216  .000081521   R-squared       =    0.2153
-------------+----------------------------------   Adj R-squared   =    0.2081
       Total |  .022440565       218  .000102938   Root MSE        =    .00903

------------------------------------------------------------------------------
         wmt |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
       mktrf |   .5885593   .0767916     7.66   0.000     .4372025     .739916
         rmw |   .2772095   .0889864     3.12   0.002     .1018167    .4526023
       _cons |  -.0005378   .0005592    -0.96   0.337    -.0016399    .0005643
-------------+----------------------------------------------------------------
         rho |  -.1082235
------------------------------------------------------------------------------
Durbin-Watson statistic (original)    2.209884
Durbin-Watson statistic (transformed) 1.996325
```

```
. prais cat mktrf hml

Iteration 0:  rho = 0.0000
Iteration 1:  rho = -0.0425
Iteration 2:  rho = -0.0434
Iteration 3:  rho = -0.0435
Iteration 4:  rho = -0.0435

Prais-Winsten AR(1) regression -- iterated estimates

      Source |       SS           df       MS      Number of obs   =       219
-------------+----------------------------------   F(2, 216)       =    118.45
       Model |  .029640473         2  .014820237   Prob > F        =    0.0000
    Residual |  .027024647       216  .000125114   R-squared       =    0.5231
-------------+----------------------------------   Adj R-squared   =    0.5187
       Total |   .05666512       218  .000259932   Root MSE        =    .01119

------------------------------------------------------------------------------
         cat |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
       mktrf |    1.104218   .0922621    11.97   0.000     .9223684    1.286067
         hml |    .8128821   .0684118    11.88   0.000      .678042    .9477222
       _cons |   -.001056   .0007339    -1.44   0.152    -.0025025    .0003904
-------------+----------------------------------------------------------------
         rho |   -.0434711
------------------------------------------------------------------------------
Durbin-Watson statistic (original)    2.083498
Durbin-Watson statistic (transformed) 1.997596


. prais de mktrf smb hml

Iteration 0:  rho = 0.0000
Iteration 1:  rho = -0.0824
Iteration 2:  rho = -0.0858
Iteration 3:  rho = -0.0860
Iteration 4:  rho = -0.0860
Iteration 5:  rho = -0.0860

Prais-Winsten AR(1) regression -- iterated estimates

      Source |       SS           df       MS      Number of obs   =       219
-------------+----------------------------------   F(3, 215)       =     56.49
       Model |  .033148184         3  .011049395   Prob > F        =    0.0000
    Residual |    .0420503       215  .000195583   R-squared       =    0.4408
-------------+----------------------------------   Adj R-squared   =    0.4330
       Total |  .075198483       218  .000344947   Root MSE        =    .01399

------------------------------------------------------------------------------
          de |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
       mktrf |    1.249646   .1247138    10.02   0.000     1.003827    1.495464
         smb |    .2413524   .1219612     1.98   0.049     .0009598    .4817451
         hml |    .5764177   .0942589     6.12   0.000     .3906279    .7622075
       _cons |   -.0003365   .0008852    -0.38   0.704    -.0020813    .0014084
-------------+----------------------------------------------------------------
         rho |   -.0859934
------------------------------------------------------------------------------
Durbin-Watson statistic (original)    2.153830
Durbin-Watson statistic (transformed) 1.994955
```

```
. prais nfg mktrf hml

Iteration 0:  rho = 0.0000
Iteration 1:  rho = -0.1280
Iteration 2:  rho = -0.1315
Iteration 3:  rho = -0.1315
Iteration 4:  rho = -0.1315
Iteration 5:  rho = -0.1315

Prais-Winsten AR(1) regression -- iterated estimates

      Source |       SS           df       MS      Number of obs   =       219
-------------+----------------------------------   F(2, 216)       =     69.72
       Model |  .016510883         2  .008255441   Prob > F        =    0.0000
    Residual |  .025577616       216  .000118415   R-squared       =    0.3923
-------------+----------------------------------   Adj R-squared   =    0.3867
       Total |  .042088499       218  .000193067   Root MSE        =    .01088

------------------------------------------------------------------------------
         nfg |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
       mktrf |   .6227529   .0894831     6.96   0.000      .446381    .7991249
         hml |   .7102385   .0661315    10.74   0.000     .5798928    .8405842
       _cons |   .0001844     .00066     0.28   0.780    -.0011165    .0014853
-------------+----------------------------------------------------------------
         rho |  -.1315467
------------------------------------------------------------------------------
Durbin-Watson statistic (original)    2.253091
Durbin-Watson statistic (transformed) 2.003545
```