Some Properties of Associates of Subsets of FSP-Points Set

Vaddiparthi Yogeswara, Biswajit Rath, Ch.Ramasanyasi Rao, and D. Raghu Ram

1Associate Professor, Dept. Mathematics, GIT GITAM University, Visakhapatnam-530045, A.P State, India
2Assistant Professor, Dept. of Applied Mathematics, GIS GITAM University, Visakhapatnam-530045, A.P State, India
3 Research Scholar : Dept. of Mathematics, GIT, GITAM University, Visakhapatnam 530045, A.P State, India
vaddiparthyy@yahoo.com; urwithbr@gmail.com; rams.mathematics@gmail.com; draghuram84@gmail.com

ABSTRACT

In this paper, based upon Fs-set theory [1], we define a crisp Fs-points set FSP(𝒜) for given Fs-set 𝒜 and establish a pair of relations between collection of all Fs-subsets of a given Fs-set 𝒜 and collection of all crisp subsets of Fs-points set FSP(𝒜) of the same Fs-set 𝒜 and prove one of the relations is a meet complete homomorphism and the other is a join complete homomorphism and search properties of relations between Fs-complemented sets and complemented constructed crisp sets via these homomorphisms.

Key word: Fs-set, Fs-subset, Fs-complement, Fs-function, Fs-point

1 Introduction:

Ever since Zadeh [17] introduced the notion of fuzzy sets in his pioneering work, several mathematicians studied numerous aspects of fuzzy sets.

Murthy [7] introduced f-sets in order to prove Axiom of choice for fuzzy sets. The following example shows why the introduction of f-set theory is necessitated. Let A be non-empty and consider a diamond lattice \(L = \{0, \alpha \parallel \beta, 1\}\). Define two fuzzy sets \(f\) and \(g\) from A into \(L\) such that \(f(x) = \alpha\) and \(g(x) = \beta\). Here both \(f\) and \(g\) are non-empty fuzzy sets. The Cartesian product of \(f\) and \(g\) from A into \(L\) is given by \((f \times g)(x) = f(x) \land g(x) = \alpha \land \beta = 0\). That is, \(f \times g\) is a empty set. Even though both \(f\) and \(g\) are non-empty fuzzy sets, their fuzzy Cartesian product is empty showing that the failure of Axiom of choice in L-fuzzy set theory [10]. The collection of all f-subsets of a given f-set with Murthy’s definition [7] f-complement [10] could not form a compete Boolean algebra. Vaddiparthi Yogeswara, G.Srinivas and Biswajit Rath introduced the concept of Fs-set and developed the theory of Fs-sets in order to prove collection of all Fs-subsets of given Fs-set is a complete Boolean algebra under Fs-unions, Fs-intersections and Fs-complements. The Fs-sets they introduced contain Boolean valued membership functions. They are successful in their efforts in proving that result with some conditions. In papers [2] and [3] Vaddiparthi Yogeswara, Biswajit Rath and S.V.G.Reddy introduced the concept of Fs-Function between two Fs-subsets of given Fs-set and defined an image of an Fs-subset under a given Fs-function. Also they studied the properties of images under various kinds of Fs-functions.

DOI: 10.14738/tmlai.46.2300
Publication Date: 12th November, 2016
URL: http://dx.doi.org/10.14738/tmlai.46.2300
In this paper, we construct a crisp set FSP(𝒜) of all Fs-points of given Fs-set 𝒜 such that there is a pair of relations between collection of all Fs-subsets of 𝒜 and collection of all crisp sub sets of FSP(𝒜), such that one of the relations is a complete meet homomorphism and other is a complete join homomorphism. Here the operations on collection of Fs-subsets of 𝒜 are Fs-union, Fs-intersection and Fs-complement. The operations on FSP(𝒜) of are usual crisp set union, crisp set intersection and crisp set complement. The correspondences between them are denoted by the same symbol ‘~’ in the later contexts. The detailed definitions of Fs-point and FSP (𝒜) for given Fs-set 𝒜 are discussed before defining those relations mentioned above. For smooth reading of paper, the theory of Fs-sets and Fs-functions in brief is dealt with in first two sections. We denote the largest element of a complete Boolean algebra $L_A$ [1.1] by $M_A$ or 1. We denote Fs-union and crisp set union by same symbol $\cup$ and similarly Fs-intersection and crisp set intersection by the same symbol $\cap$. For all lattice theoretic properties and Boolean algebraic properties one can refer Szasz [13], Garret Birkhoff[14], Steven Givant • Paul Halmos[12] and Thomas Jech[15]

2 Fs-Sets

2.1 Definition

Let $U$ be a universal set, $A_1 \subseteq U$ and let $A \subseteq U$ be non-empty. A four tuple $𝒜 = (A_1, A, \overline{A} (\mu_{1A_1}, \mu_{2A}), L_A)$ is said be an Fs-set if, and only if

1. $A \subseteq A_1$
2. $L_A$ is a complete Boolean Algebra
3. $\mu_{1A_1}: A_1 \rightarrow L_A , \mu_{2A}: A \rightarrow L_A$ are functions such that $\mu_{1A_1}|A \geq \mu_{2A}$
4. $\overline{A}: A \rightarrow L_A$ is defined by $\overline{A} x = \mu_{1A_1} x \land (\mu_{2A} x)^c$, for each $x \in A$

2.2 Definition:

Fs-subset

Let $𝒜= (A_1, A, \overline{A} (\mu_{1A_1}, \mu_{2A}), L_A)$ and $ℬ= (B_1, B, \overline{B} (\mu_{1B_1}, \mu_{2B}), L_B)$ be a pair of Fs-sets. $ℬ$ is said to be an Fs-subset of $𝒜$, denoted by $ℬ \subseteq 𝒜$, if, and only if

1. $B_1 \subseteq A_1 , A \subseteq B$
2. $L_B$ is a complete subalgebra of $L_A$ or $L_B \leq L_A$
3. $\mu_{1B_1} \leq \mu_{1A_1}|B_1$, and $\mu_{2B}|A \geq \mu_{2A}$

2.3 Proposition:

Let $ℬ$ and $𝒜$ be a pair of Fs-sets such that $ℬ \subseteq 𝒜$. Then $\overline{ℬ} x \leq \overline{A} x$ is true for each $x \in A$

2.3.1 Remark:

For some $L_X$, such that $L_X \leq L_A$ a four tuple $\chi = (X_1, X, \overline{X} (\mu_{1X_1}, \mu_{2X}), L_X)$ is not an Fs-set if, and only if

(a) $X \not\subseteq X_1$ or
(b) $\mu_{1X_1} x \geq \mu_{2X} x$, for some $x \in X \cap X_1$

URL: http://dx.doi.org/10.14738/tmlai.46.2300
Here onwards, any object of this type is called an Fs-empty set of first kind and we accept that it is an Fs-subset of \( \mathcal{B} \) for any \( \mathcal{B} \subseteq \mathcal{A} \).

### 2.4 Definition:

An Fs-subset \( \mathcal{Y} = \left( Y_1, Y, \overline{Y} \left( \mu_{1Y_1}, \mu_{2Y} \right), L_Y \right) \) of \( \mathcal{A} \), is said to be an Fs-empty set of second kind if, and only if

\[
\begin{align*}
(a') & \quad Y_1 = Y \\
(b') & \quad L_Y \leq L_A \\
(c') & \quad \overline{Y} = 0
\end{align*}
\]

#### 2.4.1 Remark:

We denote Fs-empty set of first kind or Fs-empty set of second kind by \( \Phi \).

### 2.5 Definition:

Let \( \mathcal{B}_1 = (B_{11}, B_1, \overline{B}_1, L_{B_1}) \) and \( \mathcal{B}_2 = (B_{12}, B_2, \overline{B}_2, L_{B_2}) \) be a pair of Fs-sets. We say that \( \mathcal{B}_1 \) and \( \mathcal{B}_2 \) are equal, denoted by \( \mathcal{B}_1 = \mathcal{B}_2 \) if, only if

\[
\begin{align*}
(1) & \quad B_{11} = B_{12}, \quad B_1 = B_2 \\
(2) & \quad L_{B_1} = L_{B_2} \\
(3) & \quad \text{(a) } \mu_{1B_{11}} = \mu_{1B_{12}} \text{ and } \mu_{2B_1} = \mu_{2B_2} \text{, or (b) } \overline{B}_1 = \overline{B}_2
\end{align*}
\]

#### 2.5.1 Remark:

We can easily observed that 3(a) and 3(b) not equivalent statements.

### 2.6 Proposition:

\( \mathcal{B}_1 = (B_{11}, B_1, \overline{B}_1, L_{B_1}) \) and \( \mathcal{B}_2 = (B_{12}, B_2, \overline{B}_2, L_{B_2}) \) are equal if, only if \( \mathcal{B}_1 \subseteq \mathcal{B}_2 \) and \( \mathcal{B}_2 \subseteq \mathcal{B}_1 \).

### 2.7 Definition of Fs-union for a given pair of Fs-subsets of \( \mathcal{A} \):

Let \( \mathcal{B} = (B_1, B, \overline{B}(\mu_{1B_1}, \mu_{2B}), L_B) \) and \( \mathcal{C} = (C_1, C, \overline{C}(\mu_{1C_1}, \mu_{2C}), L_C) \), be a pair of Fs-subsets of \( \mathcal{A} \). Then, the Fs-union of \( \mathcal{B} \) and \( \mathcal{C} \), denoted by \( \mathcal{B} \cup \mathcal{C} \), is defined as

\[
\mathcal{B} \cup \mathcal{C} = \mathcal{D} = (D_1, D, \overline{D}(\mu_{1D_1}, \mu_{2D}), L_D), \text{ where}
\]

\[
\begin{align*}
(1) & \quad D_1 = B_1 \cup C_1, \quad D = B \cap C \\
(2) & \quad L_D = L_B \lor L_C = \text{complete subalgebra generated by } L_B \cup L_C \\
(3) & \quad \mu_{1D_1}: D_1 \rightarrow L_D \text{ is defined by } \mu_{1D_1}x = (\mu_{1B_1} \lor \mu_{1C_1})x \\
& \quad \mu_{2D}: D \rightarrow L_D \text{ is defined by } \mu_{2D}x = \mu_{2B}x \land \mu_{2C}x \\
& \quad \overline{D}: D \rightarrow L_D \text{ is defined by } \overline{D}x = \mu_{1D_1}x \land (\mu_{2D}x)^c
\end{align*}
\]

### 2.8 Proposition:

\( \mathcal{B} \cup \mathcal{C} \) is an Fs-subset of \( \mathcal{A} \).
2.9 Definition of Fs-intersection for a given pair of Fs-subsets of $\mathcal{A}$:

Let $\mathcal{B} = (B_1, B, \overline{B}(\mu_{1B_1}, \mu_{2B}), L_B)$ and $\mathcal{C} = (C_1, C, \overline{C}(\mu_{1C_1}, \mu_{2C}), L_C)$ be a pair of Fs-subsets of $\mathcal{A}$ satisfying the following conditions:

(i) $B_1 \cap C_1 \supseteq B \cup C$

(ii) $\mu_{1B_1}x \land \mu_{1C_1}x \geq (\mu_{2B} \lor \mu_{2C})x$, for each $x \in A$

Then, the Fs-intersection of $\mathcal{B}$ and $\mathcal{C}$, denoted by $\mathcal{B} \cap \mathcal{C}$ is defined as

$$\mathcal{B} \cap \mathcal{C} = \overline{E}(E_1, E, \overline{E}(\mu_{1E_1}, \mu_{2E}), L_E),$$

where

(a) $E_1 = B_1 \cap C_1$, $E = B \cup C$

(b) $L_E = L_B \land L_C = L_B \land L_C$

(c) $\mu_{1E_1} : E_1 \rightarrow L_E$ is defined by $\mu_{1E_1}x = \mu_{1B_1}x \land \mu_{1C_1}x$

$\mu_{2E} : E \rightarrow L_E$ is defined by

$\mu_{2E}x = (\mu_{2B} \lor \mu_{2C})x$

$\overline{E} : E \rightarrow L_E$ is defined by

$\overline{E}x = \mu_{1E_1}x \land (\mu_{2E}x)^c$.

2.9.1 Remark:

If (i) or (ii) fails we define $\mathcal{B} \cap \mathcal{C}$ as $\mathcal{B} \cap \mathcal{C} = \Phi_{\mathcal{A}}$, which is the Fs-empty set of first kind.

2.10 Proposition:

For any Fs-subsets $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ of $\mathcal{A} = (A_1, A, A(\mu_{1A_1}, \mu_{2A}), L_A)$, the following associative laws are true:

(I) $\mathcal{B} \cup (\mathcal{C} \cup \mathcal{D}) = (\mathcal{B} \cup \mathcal{C}) \cup \mathcal{D}$

(II) $\mathcal{B} \cap (\mathcal{C} \cap \mathcal{D}) = (\mathcal{B} \cap \mathcal{C}) \cap \mathcal{D}$, whenever Fs-intersections exist.

2.11 Arbitrary Fs-unions and arbitrary Fs-intersections:

Given a family $\mathcal{B}_i \in \mathcal{I}$ of Fs-subsets of $\mathcal{A} = (A_1, A, A(\mu_{1A_1}, \mu_{2A}), L_A)$, where

$\mathcal{B}_i = (B_{i_1}, B_i, \overline{B}_i(\mu_{1B_{i_1}}, \mu_{2B_i}), L_{B_i})$, for any $i \in I$

2.12 Definition of Fs-union is as follows

Case (1): For $I = \Phi$, define Fs-union of $\mathcal{(B)_i}$, denoted by $\bigcup_{i \in I} B_i$ as $\bigcup_{i \in I} B_i = \Phi_{\mathcal{A}}$, which is the Fs-empty set.

Case (2): Define for $I \neq \Phi$, Fs-union of $\mathcal{(B)_i}$ denoted by $\bigcup_{i \in I} B_i$ as follow

$$\bigcup_{i \in I} B_i = B = (B_1, B, \overline{B}(\mu_{1B_1}, \mu_{2B}), L_B),$$

where

(a) $B_1 = \bigcup_{i \in I} B_{i_1}, B = \bigcap_{i \in I} B_i$

(b) $L_B = V_{i \in I} L_{B_i} =$ complete subalgebra generated by $U_{i \in I} (L_i = L_{B_i})$

(c) $\mu_{1B_1} : B_1 \rightarrow L_B$ is defined by

$\mu_{1B_1}x = (V_{i \in I} \mu_{1B_{i_1}}x = V_{i \in I} \mu_{1B_{i_1}}x$, where

$L_i = \{i \in I| x \in B_i\}$

$\mu_{2B} : B \rightarrow L_B$ is defined by $\mu_{2B}x = (\bigwedge_{i \in I} \mu_{2B_i})x$

= $\bigwedge_{i \in I} \mu_{2B_i}x$.

URL: http://dx.doi.org/10.14738/tmmlai.46.2300
\( \mathcal{B} : \mathcal{B} \rightarrow L_{\mathcal{B}} \) is defined by \( \mathcal{B}x = \mu_{1\mathcal{B}}x \land (\mu_{2\mathcal{B}}x)^c \)

### 2.12.1 Remark

We can easily show that (d) \( B_1 \supseteq \mathcal{B} \) and \( \mu_{1\mathcal{B}}|\mathcal{B} \geq \mu_{2\mathcal{B}} \).

### 2.13 Definition of Fs-intersection:

Case (1): For \( I = \Phi \), we define Fs-intersection of \( (\mathcal{B}_i)_{i \in I} \), denoted by \( \bigcap_{i \in I} \mathcal{B}_i \), as \( \bigcap_{i \in I} \mathcal{B}_i = \mathcal{A} \).

Case (2): Suppose \( \bigcap_{i \in I} B_1 \supseteq U_{i \in I} B_1 \) and \( \land_{i \in I} \mu_{1\mathcal{B}_i}(U_{i \in I} B_i) \geq V_{i \in I} \mu_{2\mathcal{B}_i} \).

Then, we define Fs-intersection of \( (\mathcal{B}_i)_{i \in I} \), denoted by \( \bigcap_{i \in I} \mathcal{B}_i \), as follows

\[
\bigcap_{i \in I} \mathcal{B}_i = C = (C_1, C, \bar{C}(\mu_{1C_1}, \mu_{2C}), L_C)
\]

(a') \( C_1 = \bigcap_{i \in I} B_1 \), \( C = \bigcup_{i \in I} B_i \)

(b') \( L_C = \land_{i \in I} L_{\mathcal{B}_i} \)

(c) \( \mu_{1C_1} : C_1 \rightarrow L_C \) is defined by \( \mu_{1C_1}x = (\land_{i \in I} \mu_{1\mathcal{B}_i})x = \land_{i \in I} \mu_{1\mathcal{B}_i}x \)

\( \mu_{2C} : C \rightarrow L_C \) is defined by \( \mu_{2C}x = (\lor_{i \in I} \mu_{2\mathcal{B}_i})x = \lor_{i \in I} \mu_{2\mathcal{B}_i}x \),

where, \( I_x = \{ i \in I \mid x \in \mathcal{B}_i \} \)

\( \bar{C} : C \rightarrow L_C \) is defined by \( \bar{C}x = \mu_{1C_1}x \land (\mu_{2C}x)^c \).

Case (3): \( \bigcap_{i \in I} B_1 \nsubseteq U_{i \in I} B_i \) or \( \land_{i \in I} \mu_{1\mathcal{B}_i}(U_{i \in I} B_i) \nsubseteq V_{i \in I} \mu_{2\mathcal{B}_i} \).

We define

\[
\bigcap_{i \in I} \mathcal{B}_i = \Phi_{\mathcal{A}}
\]

### 2.13.1 Lemma:

For any Fs-subset \( \mathcal{B} = (B_1, B, \bar{B}(\mu_{1\mathcal{B}}, \mu_{2\mathcal{B}}), L_{\mathcal{B}}) \) and \( \mathcal{B} \subseteq \mathcal{B}_1 = (B_1, B_1, \bar{B}_1(\mu_{1\mathcal{B}_1}, \mu_{2\mathcal{B}_1}), L_{\mathcal{B}_1}) \) for each \( i \in I \), \( \bigcap_{i \in I} \mathcal{B}_i \) exists and \( \mathcal{B} \subseteq \bigcap_{i \in I} \mathcal{B}_i \).

### 2.14 Proposition:

\( (\mathcal{L}(\mathcal{A}), \land) \) is \( \land \)-complete lattices.

### 2.14.1 Corollary:

For any Fs-subset \( \mathcal{B} \) of \( \mathcal{A} \), the following results are true

(i) \( \Phi_{\mathcal{A}} \cup \mathcal{B} = \mathcal{B} \)

(ii) \( \Phi_{\mathcal{A}} \cap \mathcal{B} = \Phi_{\mathcal{A}} \).

### 2.15 Proposition:

\( (\mathcal{L}(\mathcal{A}), \lor) \) is \( \lor \)-complete lattices.

### 2.15.1 Corollary:

\( (\mathcal{L}(\mathcal{A}), \lor, \land) \) is a complete lattice with \( \lor \) and \( \land \)
2.16 Proposition:
Let $\mathcal{B} = (B_1, B_2, B_{(1B_1, 1B_2)}, L_B)$, $\mathcal{C} = (C, C, C(\mu_{1C_1}, \mu_{2C}), L_C)$ and $\mathcal{D} = (D_1, D_2, D_{(1D_1, 1D_2)}, L_D)$. Then $\mathcal{B} \cup (\mathcal{C} \cap \mathcal{D}) = (\mathcal{B} \cup \mathcal{C}) \cap (\mathcal{B} \cup \mathcal{D})$ provided $\mathcal{C} \cap \mathcal{D}$ exists.

2.17 Proposition:
Let $\mathcal{B} = (B_1, B_2, B_{(1B_1, 1B_2)}, L_B)$, $\mathcal{C} = (C, C, C(\mu_{1C_1}, \mu_{2C}), L_C)$ and $\mathcal{D} = (D_1, D_2, D_{(1D_1, 1D_2)}, L_D)$. Then $\mathcal{B} \cap (\mathcal{C} \cup \mathcal{D}) = (\mathcal{B} \cap \mathcal{C}) \cup (B \cap \mathcal{D})$ provided in R.H.S $(\mathcal{B} \cap \mathcal{C})$ and $(B \cap \mathcal{D})$ exists.

2.18 Definition of Fs-complement of an Fs-subset:
Consider a particular Fs-set $\mathcal{A} = (A_1, A_2, \bar{A} (\mu_{1A_1}, \mu_{2A}), L_A)$, $\mathcal{A} \neq \Phi$, where

(i) $A \subseteq A_1$
(ii) $L_A = [0, M_A]$, $M_A \leftarrow \bar{A}A = \cup_{a \in A} \bar{A} a$
(iii) $\mu_{1A_1} = M_A$, $\mu_{2A} = 0$,

$\bar{A}x = \mu_{1A_1}x \land (\mu_{2A}x)^c = M_A$, for each $x \in A$

Given $\mathcal{B} = (B_1, B_2, B_{(1B_1, 1B_2)}, L_B)$. We define Fs-complement of $\mathcal{B}$, denoted by $\mathcal{B}^{C,\mathcal{A}}$ for $B=A$ and $L_B = L_A$ as follows:

$\mathcal{B}^{C,\mathcal{A}} = D = (D_1, D_2, D_{(1D_1, 1D_2)}, L_D)$, where

(a') $D_1 = C_1 \cup B_1 = B_1^c \cup A$, $D = B = A$
(b') $L_D = L_A$
(c') $\mu_{1D_1}: D_1 \rightarrow L_A$, is defined by $\mu_{1D_1}x = M_A$

$\mu_{2D_2}: A \rightarrow L_A$, is defined by $\mu_{2D_2}x = \bar{B}x = \mu_{1B_1}x \land (\mu_{2B}x)^c$

$\bar{D}: A \rightarrow L_A$, is defined by $\bar{D}x = \mu_{1D_1}x \land (\mu_{2D_2}x)^c = M_A \land (\bar{B}x)^c = (Bx)^c$.

2.19 Proposition:
$\mathcal{A}^{C,\mathcal{A}} = \Phi_{\mathcal{A}}$

2.20 Definition:
Define $(\Phi_{\mathcal{A}})^{C,\mathcal{A}} = \mathcal{A}$

2.21 Proposition:
For $\mathcal{B} = (B_1, B_2, B_{(1B_1, 1B_2)}, L_B)$, $\mathcal{C} = (C_1, C, C(\mu_{1C_1}, \mu_{2C}), L_C)$, which are non Fs-empty sets and $B = C = A$, $L_B = L_C = L_A$

(1) $\mathcal{B} \cap \mathcal{B}^{C,\mathcal{A}} = \Phi_{\mathcal{A}}$
(2) $\mathcal{B} \cup \mathcal{B}^{C,\mathcal{A}} = \mathcal{A}$
(3) $(\mathcal{B}^{C,\mathcal{A}})^{C,\mathcal{A}} = \mathcal{B}$
(4) $\mathcal{B} \subseteq \mathcal{C}$ if and only if $\mathcal{C}^{C,\mathcal{A}} \subseteq \mathcal{B}^{C,\mathcal{A}}$

2.22 Proposition:
Fs-De-Morgan’s laws for a given pair of Fs-subsets:

URL: [http://dx.doi.org/10.14738/tmlai.46.2300](http://dx.doi.org/10.14738/tmlai.46.2300)
For any pair of Fs-sets $B=(B_1, B, \overline{B} \mu_{1B}, \mu_{2B}), B_B$ and $C=(C_1, C, \overline{C} \mu_{1C}, \mu_{2C}), B_C$, with $B = C = A$ and $L_B = L_C = L_A$, we will have

(i) $(B \cup C)^C = B^C \cup C^C$ if $(\overline{B} x) \land (\overline{C} x) \leq (\mu_{1B} x)^C \lor (\mu_{2C} x) \land (\mu_{1C} x)^C \lor (\mu_{2B} x)$, for each $x \in A$

(ii) $(B \cap C)^C = B^C \cup C^C$, whenever $B \cap C$ exists.

### 2.23 Fs-De Morgan laws for any given arbitrary family of Fs-sets:

**Proposition:** Given a family of Fs-subsets $(B_i)_{i=1}^n$ of $\mathcal{A} = (A_1, A, \overline{A} \mu_{1A}, \mu_{2A}, L_A)$, where $L_A = [0, M_A], \mu_{1A} = M_A, \mu_{2A} = 0, \overline{A} x = M_A$

(i) $(\bigcap_{i \in I} B_i)^C = \bigcap_{i \in I} B_i^C$, for $I \neq \emptyset$, where $B_i = (B_{i1}, B_i, \overline{B}_i \mu_{1B}, \mu_{2B}) \cup B_B$ and

1. $B_i = A, L_{B_i} = L_A$ provided $\forall_{i \in I}[B_i(B_i x)] \leq \bigwedge_{i \neq j}(\mu_{1B_i} x)^C \lor \mu_{2B_j} x$

(ii) $(\bigcap_{i \in I} B_i)^C = \bigcup_{i \in I} B_i^C$, whenever $\bigcap_{i \in I} B_i$ exist

### 3 Fs-point

#### 3.1 Definition

We define an object, for $b \in A, \beta \in L_A$ such that $\beta \leq \overline{A} b$ - denoted by $(b, \beta)$ as follows

$$(b, \beta) = (B_1, B, \overline{B} \mu_{1B}, \mu_{2B}), (B_B)$$, where $A \subseteq B \subseteq B_1 \subseteq A_1, L_B \subseteq L_A$, such that $\mu_{1B} x, \mu_{2B} x \in L_B, \alpha \leq \mu_{1A} x, \forall x \in A, \beta \in L_A$

$\mu_{1B} = \begin{cases} \mu_{2A} x, & x \neq b, x \in A \\ \beta \lor \mu_{2A} b, & x = b \\ \alpha, & x \in A, x \in A_1 \end{cases}$

And $\mu_{2B} x = \begin{cases} \mu_{2A} x, & x \in A \\ \alpha, & x \in A, x \in B \end{cases}$

#### 3.2 Lemma:

(a) $\beta \leq \mu_{1A} b$ and $\beta \leq (\mu_{2A} b)^C$

(b) $\mu_{1B} b \geq \mu_{2B} b$

(c) $\mu_{1B} b \leq \mu_{1A} b$

(d) $\mu_{2B} b \geq \mu_{2A} b$

(e) $\overline{B} b = \beta$

(f) $(b, \beta)$ is Fs-subset of $\mathcal{A}$

**Proof:**

(a): Given $\beta \leq \overline{A} b = \mu_{1A} b \land (\mu_{2A} b)^C$

$\Rightarrow \beta \leq \mu_{1A} b$ and $\beta \leq (\mu_{2A} b)^C$

(b): $\mu_{1B} b \land \mu_{2B} b = (\beta \lor \mu_{2A} b) \land \mu_{2A} b = \mu_{2A} b = \mu_{2B} b$

$\Rightarrow \mu_{1B} b \geq \mu_{2B} b$

(c): $\mu_{1B} b \land \mu_{1A} b = (\beta \lor \mu_{2A} b) \land \mu_{1A} b = (\beta \land \mu_{1A} b) \lor (\mu_{1B} b \land \mu_{2A} b) = \beta \lor \mu_{2A} b = \mu_{1B} b$

$\Rightarrow \mu_{1B} b \leq \mu_{1A} b$

(d): $\mu_{2B} b \geq \mu_{2A} b$ (since $\mu_{2B} x = \mu_{2A} x, \forall x \in A$)

(e): $\overline{B} b = \mu_{1B} b \land (\mu_{2A} b)^C$

$= (\beta \lor \mu_{2A} b) \land (\mu_{2A} b)^C$

$= (\beta \land (\mu_{2A} b)^C) \lor (\mu_{2A} b \land (\mu_{2A} b)^C)$
\[(\beta \land (\mu_2 A)^c) \lor 0 \]

\[= \beta \land (\mu_2 A)^c = \beta \]

(f): Given \((b, \beta) = (B_1, B, B(\mu_{1B1}, \mu_{2B}), L_B)\)

(i) \(B_1 \subseteq A_1, A \subseteq B\)

(ii) \(L_B \subseteq L_A\)

(iii) \(\mu_{1B1} x = \begin{cases} 
\mu_{2A} x, & x \neq b, x \in A \\
\beta \lor \mu_{2A} b, & x = b
\end{cases}
\]

And \(\mu_{2B} x = \begin{cases} 
\mu_{2A} x, & x \in A \\
\alpha, & x \notin A, x \in B
\end{cases}
\]

\[\mu_{1B1} x = \mu_{2A} x = \mu_{2B} x \leq \mu_{1A1}, x \neq b, x \in A
\]

\[\mu_{1B1} b = \beta \lor \mu_{2A} b \geq \mu_{2B} b = \mu_{1B1} b \leq \mu_{1A1} b
\]

\[\therefore \mu_{1B1} x \geq \mu_{2B} x, \forall x \in B, \mu_{1B1} x \leq \mu_{1A1} x, \forall x \in B_1 \text{ and } \mu_{2B} x = \mu_{2A} x, \forall x \in A
\]

Hence \((b, \beta) = (B_1, B, B(\mu_{1B1}, \mu_{2B}), L_B)\) is Fs-subset of \(\mathcal{A}\).

Here onward \((b, \beta)\) — which is an Fs-subset of \(\mathcal{A}\), we call a \((b, \beta)\) objects of \(\mathcal{A}\).

### 3.3 Definition of a relation between objects:

For any \((b, \beta)\) objects \(B_1 = (B_{11}, B_1, B_{1}(\mu_{1B1}, \mu_{2B}), L_{1B})\) and \(B_2 = (B_{12}, B_2, B_{2}(\mu_{1B2}, \mu_{2B2}), L_{2B})\) of \(\mathcal{A}\), we say that \(B_1 R(b, \beta) B_2\) if, and only if \(\mu_{1B1} x = \mu_{2B} x, x \neq b\) and \(\forall x \in B_1\) and \(\mu_{1B1} x = \mu_{2B2} x, x \neq b\) and \(\forall x \in B_2\) and \(\mu_{1B1} b = \mu_{1B2} b = \beta \lor \mu_{2A} b\) and \(\mu_{2B} b = \mu_{2B2} b = \mu_{2A} b\).

### 3.4 Theorem:

\(R(b, \beta)\) is an equivalence relation.

Proof: The proof follows clearly from definition.

### 3.5 Definition of Fs-point:

The equivalence class corresponding to \(R(b, \beta)\) is denoted by \(\chi_b^\beta\) or \((b, \beta)\). We define this \(\chi_b^\beta\) is an Fs point of \(\mathcal{A}\).

Set of all Fs-point of \(\mathcal{A}\) is denoted by FSP(\(\mathcal{A}\)).

### 3.6 Definition:

Let \(G \subseteq \text{FSP}(\mathcal{A})\).

(a) \(G\) is said to be closed under stalks if, and only if \(\chi_b^\beta \in G, \alpha \leq \beta \Rightarrow \chi_b^\alpha \in G\)

(b) \(G\) is said to be closed under supremums if and only if \(M \subseteq L_A, \chi_b^\beta \in G, \forall \beta \in M \Rightarrow \chi_b^{\text{VM}} \in G\)

(c) \(G\) is said to be S-closed if, and only if \(G\) is closed under both stalks and supremums.

### 3.7 Theorem:

Arbitrary intersection of S-closed subset is S-closed.

### 3.8 Definition:

Let \(G \subseteq \text{FSP}(\mathcal{A})\).
Define $G^\sim = \Phi_{A}$ if $G = \Phi$. Otherwise $G^\sim = \bigcup_{x_b \in G} X_b^\beta$.

Define $B = (B_1, B, \bar{B}(\mu_1 B_1, \mu_2 B_2), L_B)$, where

$B_1 \supset B = \{b \mid x_b^\beta \in G\}$, $L_B = \bigvee_{x_b^\beta \in G} L_{\beta}$, $\mu_{1B_1} b = \bigvee_{x_b^\beta \in G} (\beta \lor \mu_{2B} b)$, $\mu_{2B} b = \mu_{2A} b$.

$\bar{B} b = \mu_{1B_1} b \land (\mu_{2B} b)^c$

$= \bigvee_{x_b^\beta \in G} (\beta \lor \mu_{2B} b) \land (\mu_{2A} b)^c$

$= \left[ \left( \bigvee_{x_b^\beta \in G} \beta \right) \lor \mu_{2A} b \right] \land (\mu_{2A} b)^c$

$= \bigvee_{x_b^\beta \in G} (\beta \land (\mu_{2A} b)^c) \lor 0$

$= \bigvee_{x_b^\beta \in G} (\beta \land (\mu_{2A} b)^c) = \bigvee_{x_b^\beta \in G} \beta$.

3.9 Theorem:

$G^\sim = B$.

Proof: Let $x_b^\beta = (B_1, B, \bar{X}(\mu_1 X_1, \mu_2 X), L_X)$, where $\beta \leq \bar{A} b$

$\mu_{1X_1} x = \begin{cases} 
\mu_{2A} X_1, & x \neq b, x \in A \\
\beta \lor \mu_{2A} b, & x = b \end{cases}$

And $\mu_{2X} x = \begin{cases} 
\mu_{2A} X, & x \in A \\
\alpha, & x \notin A, x \in B_1
\end{cases}$

Let $\bigcup_{x_b^\beta \in G} X_b^\beta = C = (C_1, C, \bar{C}(\mu_1 C_1, \mu_2 C), L_C)$, where

(I) $C_1 = B_1, C = B = \{b \mid x_b^\beta \in G\}, C_1 \supset C$.

(II) $L_C = L_X = \bigvee_{x_b^\beta \in G} L_{ \beta}$.

(III) For $b \in A$, $\mu_{1C_1} b = \bigvee_{x_b^\beta \in G} (\beta \lor \mu_{2A} b) = \mu_{1B_1} b$, $\mu_{2C} b = \mu_{2A} b = \mu_{2B} b$.

Hence $G^\sim = B$.

3.10 Definition:

For any $B \subseteq A$.

Define $B^\sim = \Phi$ if $B = \Phi_{\bar{A}}$

Let $B = (B_1, B, \bar{B}(\mu_{1B_1}, \mu_{2B_2}), L_B)$ and $B \neq \Phi_{\bar{A}}$

Define $B^\sim = \{x_b^\beta \mid b \in B, \beta \in L_B, \beta \leq \bar{B} b\}$.

3.11 Theorem:

$A = \bigcup_{x_b^\beta \in FSP(A)} X_b^\beta$

Proof: The proof follows similar lines of the proof of 4.9.
3.12 Lemma:
\( \mathcal{A}^{-} = \text{FSP}(\mathcal{A}) \)
Clearly \( \mathcal{A}^{-} \subseteq \text{FSP}(\mathcal{A}) \)
Let \( \chi^b_\beta = B = (B_1, B, \overline{B}(\mu_{1B}, \mu_{2B}), L_B) \in \text{FSP}(\mathcal{A}) \)
\( \therefore \chi^b_\beta \) is a \((b, \beta)\) object
i.e. \( b \in A, \beta \in L_A, A \subseteq B  \subseteq B_1 \subseteq A_1 \), such that \( \mu_{1B_1}x, \mu_{2B}x \in L_B, L_B \leq L_A, \forall x \in A_1, \beta \in L_A \)
\( \mu_{1B_1}x = \begin{cases} 
\mu_{2A}x, & x \neq b, x \in A \\
\beta \lor \mu_{2A}b, & x = b 
\end{cases} \)
And \( \mu_{2B}x = \begin{cases} 
\mu_{2A}x, & x \in A \\
\alpha, & x \notin A, x \in B 
\end{cases} \)
Clearly \( b \in A \subseteq B, \beta \in L_B \) and \( L_B \leq L_A \)
Hence \( \text{FSP}(\mathcal{A}) \subseteq \mathcal{A}^{-} \)
Hence \( \mathcal{A}^{-} = \text{FSP}(\mathcal{A}) \)

3.13 Theorem:
\( \mathcal{B}^{-} \) is \( S \)-closed.
Proof: Let \( \chi^b_\beta \in \mathcal{B}^{-}, \) then \( b \in B, \beta \in L_B, \beta \leq \overline{B}b \)
Let \( \delta \leq \beta, \delta \in L_B, \delta \leq \overline{B}b \)
\( \therefore \chi^b_\delta \in \mathcal{B}^{-} \)
Hence \( \mathcal{B}^{-} \) is closed under stalks.
Let \( \chi^{b_1}_{\beta_i} \in \mathcal{B}^{-} \) for \( i \in I \) then \( b \in B, \beta_i \in L_B, \beta_i \leq \overline{B}b \)
\( \Rightarrow b \in B, \lor_{i \in I} \beta_i \in L_B, \lor_{i \in I} \beta_i \leq \overline{B}b \)
\( \Rightarrow \chi^{b_1}_{\lor_{i \in I} \beta_i} \in \mathcal{B}^{-} \)
Hence \( \mathcal{B}^{-} \) is closed under supremum.
\( \therefore \mathcal{B}^{-} \) is \( S \)-closed.

3.14 Theorem:
For any \( G \subseteq \text{FSP}(\mathcal{A}), G \subseteq G^{-} \)
Proof: Case (I): \( G = \Phi \Rightarrow \text{Clear} \)
Case (II): \( G \neq \Phi, \text{we have } G^{-} = \mathcal{B}^{-} = \{ \chi^b_\beta | b \in B, \beta \in L_B, \beta \leq \overline{B}b \} \)
Where \( \mathcal{B} = (B_1, B, \overline{B}(\mu_{1B}, \mu_{2B}), L_B) \), where
\( B_1 \supseteq B = \{ b | \chi^b_\beta \in G \}, L_B = \lor_{x \in G} \mu_{1B_1}b, L_B = \lor_{x \in G} (\mu_{2B}x \lor \mu_{2A}b), \mu_{2B}b = \mu_{2Ab}, \overline{B}b = \lor_{x \in G} \beta \)
Let \( \chi^b_\beta \in G \Rightarrow b \in B, \beta \in L_B, \overline{B}b = \lor_{x \in G} \beta \Rightarrow \chi^b_\beta \in \mathcal{B}^{-} = G^{-} \Rightarrow G \subseteq G^{-} \)

URL: http://dx.doi.org/10.14738/mlai.46.2300
3.15 Theorem:
Let $\mathcal{A}$ be an Fs-set. Then the following are equivalent for any $G \subseteq \text{FSP}(\mathcal{A})$

(a) $G^\sim = G$
(b) $G$ is S-closed
(c) (i) $b \in B \Rightarrow \chi_b^{\sim B} \in G$
   (ii) $b \in B, \beta \leq \overline{B}b \Rightarrow \chi_b^\beta \in G$ where $\overline{B} = G^\sim$

Proof: We have $G^\sim = \bigcup_{\chi_b^\beta \in G} \chi_b^{\sim B}$ and from 4.9, $G^\sim = \mathcal{B} = (B_1, B, \overline{B}(\mu_{1B_1}, \mu_{2B_1}), L_B)$, where

$$B_1 \supseteq \mathcal{B} = \left\{ b | \chi_b^\beta \in G \right\}, L_B = \bigvee_{\chi_b^\beta \in G} L_\beta, \mu_{1B_1} b = \bigvee_{\chi_b^\beta \in G} (\beta \lor \mu_{2B_1} b), \mu_{2B_1} b = \mu_{2A} b, \overline{B}b = \bigvee_{\chi_b^\beta \in G} \beta$$

Also we have $G^{\sim B} = \mathcal{B}^\sim = \left\{ \chi_b^\beta | b \in B, \beta \in L_B, \beta \leq \overline{B}b \right\}$

$G = G^{\sim B} = \mathcal{B}^\sim$ which is S-closed from 4.13 gives (a) $\Rightarrow$ (b)

(i) Here $G^\sim = \mathcal{B} = (B_1, B, \overline{B}(\mu_{1B_1}, \mu_{2B_1}), L_B)$, where

$$B_1 \supseteq \mathcal{B} = \left\{ b | \chi_b^\beta \in G \right\}, L_B = \bigvee_{\chi_b^\beta \in G} L_\beta, \mu_{1B_1} b = \bigvee_{\chi_b^\beta \in G} (\beta \lor \mu_{2B_1} b), \mu_{2B_1} b = \mu_{2A} b, \overline{B}b = \bigvee_{\chi_b^\beta \in G} \beta$$

For any $b \in B, \chi_b^{\sim B} \in G$ follows from $G$ is closed under supremaums.

(ii) For any $b \in B, \beta \leq \overline{B}b$, we have $\chi_b^\beta \in G$, because $\chi_b^{\sim B} \in G$ and $G$ is S-closed which gives (b) $\Rightarrow$ (c).

(c) $\Rightarrow$ (a) is clear.

3.16 Theorem:
For any $\mathcal{B}_1$ and $\mathcal{B}_2$ such that $\mathcal{B}_1 \subseteq \mathcal{B}_2 \subseteq \mathcal{A}$, $\mathcal{B}_1^\sim \subseteq \mathcal{B}_2^\sim$ provided $\mathcal{B}_1 = \mathcal{B}_2$ where $\mathcal{B}_1 = (B_{11}, B_1, \overline{B}_1(\mu_{1B_{11}}, \mu_{2B_{11}}), L_{B_1})$ and $\mathcal{B}_2 = (B_{12}, B_2, \overline{B}_2(\mu_{1B_{12}}, \mu_{2B_{12}}), L_{B_2})$

Proof: From hypotheses, we have

(1) $B_{11} \subseteq B_{12}, B_1 \supseteq B_2$
(2) $L_{B_1} \leq L_{B_2}$
(3) $\mu_{1B_{11}} \leq \mu_{1B_{12}} | B_{11}, \mu_{2B_{11}} | B_2 \geq \mu_{2B_2}$

$\chi_b^\beta \in B_1^\sim$

Implies $b \in B_1, \beta \in L_{B_1}, \beta \leq \overline{B}_1 b$ which implies

$b \in B_2, \beta \in L_{B_2}, \beta \leq \overline{B}_2 b (\therefore B_1 \subseteq B_2)$

so that $\chi_b^\beta \in B_2^\sim$

3.16.1 Corollary:
$\mathcal{B} \subseteq \mathcal{A} \Rightarrow \text{FSP}(\mathcal{B}) \subseteq \text{FSP}(\mathcal{A})$

3.17 Result:
$B_1 \subseteq B_2$ implies $B_1 \subseteq B_2 \cup B_3$ for any Fs-subset $B_3$

3.18 Result:
$\chi_b^\beta \subseteq G^\sim$ for any $\chi_b^\beta \in G$ such that $G \subseteq \text{FSP}(\mathcal{A})$. 
Proof: \( X_b^\beta \in G \) is an FPs-point of \( \mathcal{A} \) and \( G^\sim = \bigcup_{X_b^\beta \in G} X_b^\beta \) gives \( X_b^\beta \subseteq G^\sim, \) \( i \in I \) such that \( G_i \subseteq G, \)

### 3.19 Recall 1.16 for any Family \((G_i)_{i \in I}\) of Fs-subsets of \( \mathcal{A} \) such that \( G_1 \subseteq G, \bigcup_{i \in I} G_i \subseteq G. \)

### 3.20 Proposition:

\( G_1 \sim \subseteq G_2 \sim \) for any two subsets \( G_1 \) and \( G_2 \) of \( \mathcal{FSP(A)} \), such that \( G_1 \subseteq G_2 \).

Proof: The proof follows clearly if \( G_1 = \Phi \) because \( G_1 \sim = \Phi = \mathcal{L}_A \) which is an Fs-empty subset of \( \mathcal{A} \)

For \( G_1 \neq \Phi, \)

\( \chi_b^\beta \in G_1 \subseteq G_2 \) implying \( \chi_b^\beta \subseteq G_2 \sim \) from 4.18 again implying

\( U_{X_b^\beta \in G_1} \chi_b^\beta \subseteq G_2 \sim \) from 4.19 so that \( G_1 \sim \subseteq G_2 \sim \)

### 3.21 Theorem:

For any Fs-subset \( B \) of an Fs-set \( \mathcal{A}, B\sim = B. \)

Proof: For \( B = \Phi, \mathcal{F}_{\mathcal{A}} \)-Fs-empty subset of \( \mathcal{A}, B\sim = \Phi \) the crisp empty subset, \( B\sim = \Phi, B\sim = B \)

For \( B \neq \Phi, B\sim = \{X_b^{\beta}(b \in B, \beta \in L_B, \beta \leq B) \} \).

Since each \( \chi_b^\beta \) in \( B\sim \) is an Fs-subset of \( B \) it follows \( U_{X_b^\beta \in B\sim} X_b^\beta \subseteq B. \)

But \( G\sim = \bigcup_{X_b^\beta \in G} X_b^\beta \) implies \( (B\sim)^- = \bigcup_{X_b^\beta \in G\sim} X_b^\beta \)

Where \( B\sim^- = \{X_b^{\beta}(X_b^{\beta} \subseteq B\} \)

\( \therefore U_{X_b^\beta \in B\sim} X_b^\beta \subseteq B = U_{X_b^\beta \in G\sim} X_b^\beta = U_{X_b^\beta \in B\sim} X_b^\beta = (B^-)^- \)

So that \( B\sim^- = B. \)

### 3.22 Theorem:

\((B \cap C)^- = B^- \cap C^- \) for any Fs-subsets \( B = (B_1, B, \mu_1, \mu_2, L_B), \) and \( C = (C_1, C, \mu_1, \mu_2, L_C) \)

of \( \mathcal{A} \) such that \( B = C. \)

Proof: For \( B \cap C = \Phi, (B \cap C)^- = \Phi, \) which is the crisp empty set

For \( \chi_b^\beta \in B^- \cap C^- \), \( \chi_b^\beta \subseteq B \) and \( \chi_b^\beta \subseteq C \) which imply \( \chi_b^\beta \subseteq B \cap C \) again implying \( \chi_b^\beta \in (B \cap C)^- \) - a contradiction

For \( B \cap C \neq \Phi, \)

Say \( B \cap C = D = (D_1, D, \mu_1, \mu_2, L_D), \) where \( D = B = C \)

Then \( (B \cap C)^- \subseteq B^- \) and \( (B \cap C)^- \subseteq C^- \) from 4.16

Implying \( (B \cap C)^- \subseteq B^- \cap C^- \)

For \( \chi_b^\beta \in B^- \cap C^- \)

\( b \in B, \beta \in L_B, \beta \leq Bb \) and \( b \in C, \beta \in L_C, \beta \leq Cb \)

Implying \( b \in B \cap C, \beta \in L_B \wedge L_C, b \leq \mu_B \wedge \mu_C = (B \wedge C)b \) again implying \( \chi_b^\beta \in (B \cap C)^- \)

URL: [http://dx.doi.org/10.14738/tmlai.46.2300](http://dx.doi.org/10.14738/tmlai.46.2300)
So that \((B \cap C)^{-} \supseteq B^{-} \cap C^{-}\)

Hence \((B \cap C)^{-} = B^{-} \cap C^{-}\)

### 3.23 Proposition:
For any family of Fs-subset \((B_i)_{i \in I}\) of \(\mathcal{A}\), \((\bigcap_{i \in I} B_i)^{-} = \bigcap_{i \in I} B_i^{-}\) provided all \(B_i\)'s are equal for each \(i \in I\).

### 3.24 Theorem:
\((G_1 \cup G_2)^{-} = G_1^{-} \cup G_2^{-}\) for any subsets \(G_1\) and \(G_2\) of \(FSP(\mathcal{A})\).

Proof: For \(G_1 = \Phi\), we have \(G_1^{-} = \Phi^c\) and \((G_1 \cup G_2)^{-} = G_2^{-}\) and \(G_1^{-} \cup G_2^{-} = G_2^{-}\)

So that \((G_1 \cup G_2)^{-} = G_1^{-} \cup G_2^{-}\).

Suppose \(G_1\) and \(G_2\) be non-empty

Since \(G_1, G_2 \subseteq G_1^{-} \cup G_2^{-}\), so that \(G_1^{-} \cup G_2^{-} \subseteq (G_1 \cup G_2)^{-}\)

For \(\chi^B_{\beta} \subseteq (G_1 \cup G_2)^{-}\), \(\chi^B_{\beta} \in G_1 \cup G_2\) so that \(\chi^B_{\beta} \subseteq G_1^{-}\) or \(\chi^B_{\beta} \subseteq G_2^{-}\) so that \(\chi^B_{\beta} \subseteq G_1^{-} \cup G_2^{-}\) finally \((G_1 \cup G_2)^{-} \subseteq G_1^{-} \cup G_2^{-}\)

Hence \((G_1 \cup G_2)^{-} = G_1^{-} \cup G_2^{-}\)

### 3.25 Theorem:
\((U_{i \in I} G_i)^{-} = U_{i \in I} G_i^{-}\) for any family \((G_i)_{i \in I}\) of subsets of \(FSP(\mathcal{A})\).

### 3.25.1 Remark:
Observe that \(\chi^B_{c}\) is always an Fs-subset of \(B\) i.e. \(\chi^B_{c} \in B^{-}\) i.e. \(\chi^B_{c} \notin (B^{-})^c\)

### 3.26 Theorem:
For \(B = (B_1, B, \overline{B}(\mu_1 B, \mu_2 B), L_B) \subseteq \mathcal{A}\), \(B = A\) and \(L_A = L_B\),

\((B^c)^{-} \subseteq (B^{-})^c\)

Proof: Suppose \(B^c, \overline{B}(\mu_1 D, \mu_2 D), L_D)\). From 1.18

\((1)\) \(D_1 = C_A B_1 = B_1^f \cup A, D = B = A\)

\((2)\) \(L_D = L_A\)

\((3)\) \(\mu_1 D, D_1 \rightarrow L_A\), is defined by \(\mu_1 D, x = M_A\)

\(\mu_2 D, A \rightarrow L_A\), is defined by \(\mu_2 D, x = \overline{B}x = \mu_1 B, x \wedge (\mu_2 B x)^c\)

\(\overline{D} D, A \rightarrow L_A\), is defined by \(\overline{D} D, x = \mu_1 D, x \wedge (\mu_2 D, x)^c = M_A \wedge (Bx)^c = (Bx)^c\).

Then from 4.10 \((B^c)^{-} = D^{-} = \{x^A_\delta | d \in A, \delta \in L = L_D, \delta \leq \overline{B}d = (Bd)^c\} \text{ i.e. } \delta \leq \overline{B}d = 0\)

And \(B^{-} = \{x^B_\beta | b \in B = A, \beta \leq L_B = L_A \beta \leq \overline{B}b\}\) implying \((B^{-})^c = \{x^A_\gamma | x^A_\gamma \in B^{-}\}\)

\(\chi^B_{c} \in (B^c)^{-}\) implying \(\gamma \wedge \overline{B}c = 0\) which implies \(\gamma \leq \overline{B}c\) as \(\chi^B_{c} \notin B^{-}\) so that \(\chi^B_{c} \notin (B^{-})^c\)

Hence \((B^c)^{-} \subseteq (B^{-})^c\)

### 3.26.1 Example:
Let \(\mathcal{A} = (A_1, A, \overline{A}(\mu_1 A_1, \mu_2 A_1), L_A)\), where \(A_1 = \{a, b\}\), \(A = \{a\}, \mu_1 A_1 = 1, \mu_2 A_1 = 0\) and \(L_A = \{0, \alpha \parallel \beta, 1\}\)

Suppose \(B = (B_1, B, \overline{B}(\mu_1 B_1, \mu_2 B_1), L_B) \subseteq \mathcal{A}\), where \(B_1 = B = A = \{a\}, \mu_1 B_1 = \alpha, \mu_2 B_1 = 0\) and \(L_B = L_A \overline{B}a = \alpha\)
\( \mathcal{B}_{\mathcal{C},d} = \mathcal{D}(D_1, D, \overline{D}(\mu_{1D_1}, \mu_{2D}), L_D), \) where \( D_1 = A_1, D = A, \mu_{1D_1} \alpha = 1, \mu_{2D} \alpha = \alpha, \overline{D} = \beta, L_D = L_A \)

\[ \mathcal{B}^c_{\mathcal{C},d} \sim \mathcal{D}^c = \{ \chi_0^d | d \in A, \delta \in L_A = L_B, \delta \leq \overline{Dd} = (\overline{Bd})^c \} \text{ i.e } \delta \wedge \overline{Bd} = 0 \]

\[ \mathcal{B}^c = \{ x_b^\beta | b \in B = A, \beta \in L_B = L_A, \beta \leq \overline{Bb} \} \]

\[ \Rightarrow \mathcal{B}^c = \{ x_0^a, x_a^a \} \]

\[ \Rightarrow x_a^1 \in (\mathcal{B}^c)^c (\because 1 \not\in \overline{B}a = \alpha) \]

But \( x_a^1 \not\in (\mathcal{B}^c)^c \)

i.e. \( (\mathcal{B}^c)^c \not\subseteq (\mathcal{B}^c, d) \)

### 3.27 Theorem:

\((\mathcal{G}^-)^{\mathcal{C}, d} \subseteq (\mathcal{G})^c \) for any \( G \subseteq \text{FSP}(\mathcal{A}), \) where \( \mathcal{A} = (A_1, A, \overline{A}(\mu_{1A_1}, \mu_{2A}), L_A), \mu_{1A_1} = M_A, \mu_{2A} = 0 \) and \( L_A = [0, M_A]. \)

**Proof:** For any \( G \subseteq \text{FSP}(\mathcal{A}), \)

\[ \mathcal{G}^c = \bigcup_{b \in G} x_b^b \]

Let \( \mathcal{G}^c = \mathcal{B} = (B_1, B, \overline{B}(\mu_{1B}, \mu_{2B}), L_B), \) where \( B_1 \supseteq B = A = \{ b | x_b^b \in G \}, \)

\[ L_B = L_A, \mu_{1B_1}b = \overline{V}_{x_b^b \in G} (\beta \vee \mu_{2B}b) = \overline{V}_{x_b^b \in G} \beta, \mu_{2B}b = 0, \overline{Bb} = \overline{V}_{x_b^b \in G} \beta. \]

Let \( (\mathcal{B})^{\mathcal{C}, d} = \mathcal{D} = (D_1, D, \overline{D}(\mu_{1D_1}, \mu_{2D}), L_D) \) then

1. \( D_1 = A, D = B = A \)
2. \( \mathcal{L}_D = L_A \)
3. \( \mu_{1D_1} : D_1 \rightarrow L_A, \) is defined by \( \mu_{1D_1} \chi = M_A \)
   \[ \mu_{2D} : A \rightarrow L_A, \) is defined by \( \mu_{2D} \beta = \overline{Bb} = \overline{V}_{x_b^b \in G} \beta \]

\[ \mathcal{D}_A \rightarrow L_A, \) is defined by \( \mathcal{D}b = \mu_{1D_1}b \wedge (\mu_{2D}b)^c = M_A \wedge (\overline{Bb})^c = \left( \overline{V}_{x_b^b \in G} \beta \right)^c \]

Also \( (\mathcal{G}^-)^{\mathcal{C}, d} = (\mathcal{B})^{\mathcal{C}, d} = \mathcal{D} \)

Now, \( \mathcal{G}^c = \text{FSP}(\mathcal{A}) - G \)

Let \( (\mathcal{G})^c = \mathcal{E} = (E_1, E, \overline{E}(\mu_{1E}, \mu_{2E}), L_E), \) where

\[ E_1 = E = A = \{ c | x_c^c \in G \}, L_E = L_A, \mu_{1E}c = \overline{V}_{x_c^c \in G} (\gamma \vee \mu_{2E}c) = \overline{V}_{x_c^c \in G} \gamma, \mu_{2E}c = \mu_{2E}c = 0, \]

\[ \mathcal{E}c = \overline{V}_{x_c^c \in G} \gamma. \]

We prove \( (\mathcal{G}^-)^{\mathcal{C}, d} \subseteq (\mathcal{G})^c \) if \( \chi_b^{M_A} \in G \) or \( \chi_b^{M_A} \not\in G \)

If \( \chi_b^{M_A} \in G, \) then \( \mathcal{G}^c = \bigcup_{b \in G} x_b^b = \mathcal{B} = (B_1, B, \overline{B}(M_A, 0), L_A), \overline{B} = M_A \) implying \( \mathcal{G}^-)^{\mathcal{C}, d} = \mathcal{D} = (D_1, D, \overline{D}(M_A, M_A), L_D), \overline{D} = 0 \)

That is, \( (\mathcal{G}^-)^{\mathcal{C}, d} = \Phi_{\mathcal{C}, d} \subseteq (\mathcal{G})^c \)

If \( \chi_b^{M_A} \not\in G \) then \( \chi_b^{M_A} \in G \) implying \( (\mathcal{G})^c = \mathcal{E} = (E_1, E, \overline{E}(M_A, 0), L_A) \)

That is, \( (\mathcal{G})^c = \mathcal{A} \supseteq (\mathcal{G}^-)^{\mathcal{C}, d} \)

Hence, whether \( \chi_b^{M_A} \in G \) or \( \chi_b^{M_A} \not\in G, \) we have

**URL:** [http://dx.doi.org/10.14738/tmlai.46.2300](http://dx.doi.org/10.14738/tmlai.46.2300)
(G^c)^c \subseteq (G^c)^c

ACKNOWLEDGEMENTS

Nistala V.E.S. Murthy [7] the great teacher is acknowledged.

The first two authors acknowledge GITAM university management for providing facilities to do research.

REFERENCES


